Optimal Inventory Allocation under Substitutable Demand

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Abstract. When customers purchase products to meet their needs, if the preferred product is out of stock, customers may choose another product instead. That is, demand for certain products is substitutable. This research considers the problem of the optimal quantities of products to be stocked under substitutable demand in order to reduce inventory and minimize lost-sales, and finally maximize the overall expected profit. This study constructs a mathematical model for estimation of expected profit given a quantity allocation in a finite-capacity inventory system under substitutable demand. In addition, this research proposes an effective search method to find the optimal quantity allocation. Comparison is made on the total expected profit under the optimal quantity allocations with and without consideration of demand substitutability. The result shows that the total profit is higher when the effect of demand substitutability is taken into account. This research demonstrates the parameters influence decision making and profits.

Keywords: substitutable demand, inventory allocation, capacity constraint.

1. INTRODUCTION

When customers purchase products to meet their needs, if the preferred product is out of stock, customers may choose another product instead. That is, demand for certain products is substitutable. For example, customers who would like to buy black tea in the first place may end up purchasing green tea instead because their top choice, black tea, is not available. As a result, there are two groups of customers who may purchase a product. The first group of customers are those who pick this product as their top choice. The second group consists of customers who see this product as a substitute of the top choice product. Managers generally focus only on demand from customers of the first group. However, if the chance of a product to be purchased as a substitute is high, it can be beneficial to prepare more of this product as the overall demand for this product is higher when taking into account the second group of customers.

In practice, there is a limit on the storage capacity of a facility who stocks and displays products. How many units of each product to carry in the facility need to be carefully planned. It is undesirable to see some products running out of stock and other products having excess inventory at the same time. To generate more profit through sales of products, it is intuitive to give higher priorities to more profitable products. Abundant amount of high profitable products are prepared to make sure demand for those products are met. With demand substitutability, however, the effective amount of demand can be higher than expected and how to allocate quantities of various products becomes more complicated.

This paper considers quantity allocation problem with multiple products, limited storage capacity and demand substitutability. The objectives are two-fold: (1) to find the optimal quantity allocation such that expected total profit is maximized, and (2) to compare solutions and performance obtained when demand substitutability is or not taken into account. A mathematical model is constructed for the considered problem. Solution methods for solving the problem based on the mathematical model are proposed. Their performance is evaluated and compared. Numerical experiment is conducted to investigate the effect of demand substitutability on quantity allocation and corresponding profit.

Tayur et al. (1999) pointed out that customers are usually willing to accept products of different colors, sizes or brands in the same product group. If the top choice products are not available, customers tend to purchase substitute products instead of leaving empty handed. Rajaram and Tang (2008) studied how product substitution affects order quantities and profits. They showed that forming optimal policies with product substitution is complex. Nagarajan and Rajagopalan

(2008) studied the single-period problem where a fixed portion of unsatisfied customers will purchase other available products. They developed “partially decoupled” policies and showed that they perform well. Park and Yoon (2011) proposed a non-linear mathematical model for with shortages and one stage product substation. They also developed a two-phase solution method.

The rest of the paper is organized as follows. The next section introduces the considered quantity allocation problem. Section 3 presents the constructed mathematical model. Solution methods are proposed in section 4. The conducted numerical experiment and analysis are presented in section 5. The last section concludes the paper.

2. PROBLEM DESCRIPTION

We consider a retail sales system where a number of products are displayed. In the single period problem, the amount of each product to hold at the beginning of the period needs to be determined, taking into account the limited storage capacity of the system. It is assumed products cannot be replenished during the period. Demand is lost if it cannot be met. Unsold units at the end of the period are salvaged. In addition, we assume the products have substitutable demands. That is, if the product a customer would like to purchase is out of stock, there is a chance the customer would purchase another product that is available in the system.

Let α_{ij} denote the substitute probability that a customer who would like to purchase product i but settles for buying product j because product i is out of stock. Table 1 lists the substitute probabilities of the demand for three products. When a customer would like to purchase product 2 and finds out this product is unavailable. There is a 0.1 probability this customer would buy product 1, a 0.5 probability the customer would buy product 3, and a 0.4 probability the customer would leave without buying anything.

Table 1: Substitute probability of 3 products.

	Product (*j*)			
Product (*i*)	**1**	**2**	**3**	Leave
1	0	0.3	0.7	0.0
2	0.1	0	0.5	0.4
3	0.3	0.5	0	0.2

This research considers the problem of determining optimal quantities of products to be stocked under substitutable demand in order to reduce inventory and minimize lost-sales, and finally maximize the overall expected profit. The following is a list of notation used throughout the paper.

Parameters:

M	Number of product types
i, j	Product types
C	Storage capacity of the system
D_i	Demand for product i as top choice
R_i	Unit revenue of product i
V_i	Unit cost of product i
S_i	Unit salvage value of product i

Variables:

Q_i	Allocated quantity of product i
B_i	Units short of product i after demands for product i as the top choice are satisfied
I_i	Amount of inventory of product i after demands for product i as the top choice are satisfied
N_{ij} (k)	Amount of product j that are available to satisfy k customers whose top choice is product i
T_{ij}	Expected demand for product j as substitute when top choice, product i, is not available
X_i	Expected total demand for product i as substitute
Y_i	Expected total units of sales of product i as substitute
F_i	Expected units of sales of product i as top choice
E_i	Expected amount of ending inventory of product i
TP	Expected total profit

The following assumptions are made when constructing the mathematical model for the considered problem.

- The storage capacity of the system is fixed.
- One unit of any product requires a storage space of the same size.
- Each visiting customer purchases or attempts to purchase exactly one unit of a product.
- When both the top choice product and the substitute product are not available, customers leave without making any purchase.

3. MODEL CONSTRUCTION

Equation (1) states that the sum of units of all the products allocated to the system at the beginning of the period is equal to the storage capacity of the system. Equation (2) calculates the amount of unsatisfied demand for product i as the top choice product. Equation (3) determines the available amount of product i after demand for product i as the top choice product are all met.

$$C = \sum_{i=1}^{M} Q_i \tag{1}$$

$$B_i = \begin{cases} D_i - Q_i, & if \quad Q_i < D_i \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

$$I_i = \begin{cases} Q_i - D_i, & if \quad D_i < Q_i \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

Demand for product *j* can be from customers whose top choice products are not available and product *j* is their substitute product. When the top choice product is out of stock, customers may either try to purchase the substitute product or simply leave. If there are B_i customers whose top choice, product *i*, is not available. The probability that k of them will try to purchase product *j* as the substitute can be calculated by Equation (4). Equation (5) determines the number of customers whose top choice, product *i*, is not available and purchase product *j* as the substitute.

$$P_{ij}(k) = \left(\frac{B_i!}{k!\,(B_i - k)!} \right) \left(a_{ij}\right)^k \left(1 - a_{ij}\right)^{B_i - k} \tag{4}$$

$$N_{ij}(k) = \begin{cases} I_j, & if \quad I_j < k \\ k, & if \quad 0 < k \le I_j \\ 0, & if \quad k = 0 \end{cases} \tag{5}$$

By combining the above two equations, the expected demand for product *j* as substitute when top choice, product *i*, is not available can be determined by Equation (6). The expected total demand for product *i* as substitute the can be expressed as Equation (7).

$$T_{ij} = \sum_{k=0}^{B_i} P_{ij}(k) \cdot N_{ij}(k) \tag{6}$$

$$X_i = \sum_{\substack{j=1 \\ i \neq j}}^{M} T_{ji} \tag{7}$$

The expected total units of sales of product *i* as substitute can be calculated by Equation (8). When the expected demand is greater than available amount. The amount of sales is equal to the available amount. Otherwise, all the expected demand can be satisfied. Equation (9) expresses the expected units of sales of product *i* as the top choice, which is the smaller between the demand and the allocated quantity.

$$Y_i = \begin{cases} I_i, & if \quad X_i \ge I_i \\ X_i, & if \quad X_i < I_i \end{cases} \tag{8}$$

$$F_i = min\{D_i \,, Q_i\} \tag{9}$$

The sum of Y_i and F_i is the total amount of sales of product *i*. The expected amount of ending inventory of product *i* can then be stated by Equation (10).

$$E_i = Q_i - (F_i + Y_i) \tag{10}$$

Using Equations (1) to (10), expected total revenue, expected total cost and expected total salvage value at the end of the period can be determined. Equation (11) calculates the total profit of the system.

$$TP = \sum_{i=1}^{M} (F_i + Y_i) \cdot R_i - (Q_i \cdot V_i) + (E_i \cdot S_i) \tag{11}$$

4. SOLUTION METHODS

A simple way to find the optimal quantity allocation that maximizes expected total profit in the above mathematical model is conducting exhaustive search. This method evaluates every possible solutions and identifies the one that optimizes the objective value. Since the allocated quantities are discrete and the storage space of the system is limited, the number of possible quantity allocations is finite. Exhaustive search should be able to find the optimal allocation as long as the scale of the problem is not too large.

Table 2 lists the parameter setting of Example 1 with 3 products. Table 3 provides the substitute probabilities. The storage capacity is 20.

Table 2: Parameter setting of Example 1.

Product	Unit revenue	Unit cost	Demand	Unit salvage value
1	11	5	8	0.5
2	8	3	7	0.3
3	5	2	15	0.2

Table 3: Substitute probabilities of Example 1.

	Product (j)		
Product (i)	**1**	**2**	**3**
1	0	0.3	0.7
2	0.3	0	0.5
3	0.1	0.2	0

There is a total of 231 possible solutions. The optimal allocation of Example 1 is (Q_1, Q_2, Q_3) = (9, 9, 2) and the corresponding maximal profit is \$100.11. The run time of the exhaustive search is 0.444 second.

The drawback of the exhaustive search is the run time increases exponentially as the size of the problem becomes greater. With more products and larger storage space, the number of possible solutions increases rapidly. The exhaustive search becomes less efficient quickly when dealing with large scale problems.

With demand substitutability, shortages of less profitable products can be beneficial. Some customers may simple leave without buying anything due to the shortage and the sales is lost. However, if some of the unsatisfied customers turn to purchase more profitable products as substitutes, the total profit actually increases. Nevertheless, it is obvious that demand for the most profitable product as top choice should be fully met. In other words, the quantity allotted to the most profitable product should be no less than the demand for this product as top choice. With this property of the optimal solution, we do not have to examine solutions with the allocated quantity of the most profitable product less than the demand of the product, as these solutions are not optimal. A more efficient search method can be designed by utilizing this property.

Figure 1 is the process flow chart of the efficient search method on problems with 3 products when demand for the most profitable product is less than the storage capacity of the system. If demand for the most profitable product is greater than or equal to the storage capacity, the decision is obvious, filling the system with the most profitable product.

Table 4: Parameter setting of Example 2.

Product	**Unit revenue**	**Unit cost**	**Demand**	**Unit salvage value**
1	15	5	30	0.5
2	11	3	25	0.3
3	8	3	40	0.3
4	9	4	30	0.4
5	5	2	50	0.2

We apply this efficient search method on Example 1 and the same optimal solution is obtained. But only 91 solutions are needed to be examined and the run time is 0.156 seconds. To further compare the two search methods, they are to solve Example 2, an example of greater scale with 5 products and a storage capacity of 160. Table 4 contains the parameter setting and Table 5 provides the substitute probabilities.

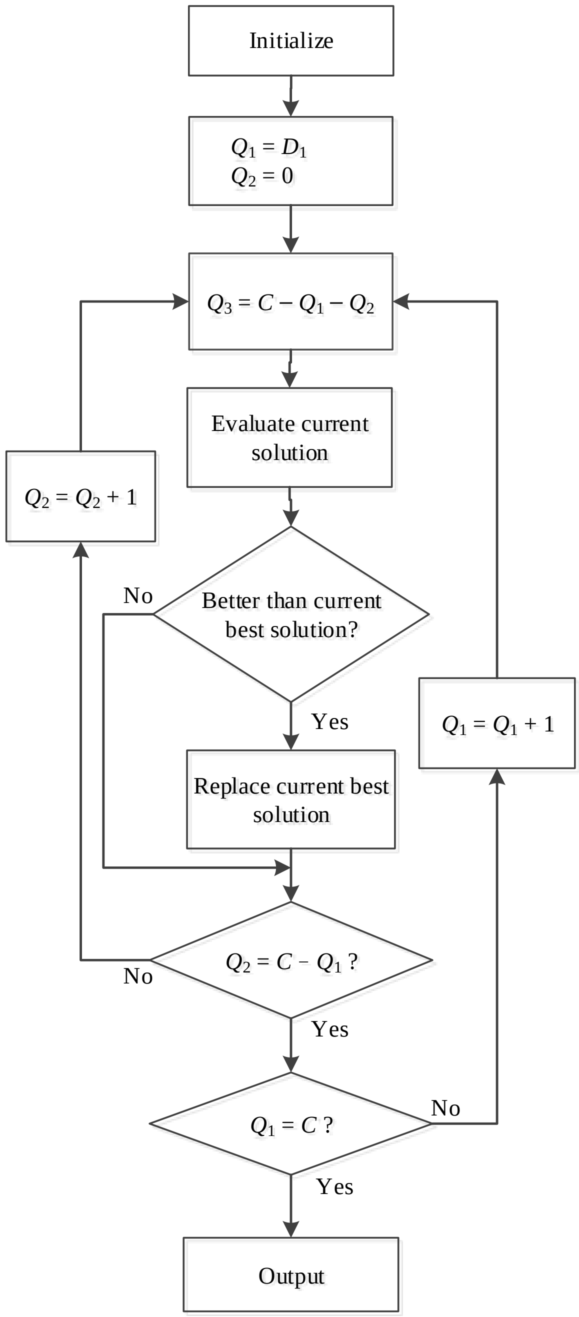

Figure 1: Process flow chart of the efficient search method

Table 5: Substitute probabilities of Example 2.

	Product (j)				
Product (i)	**1**	**2**	**3**	**4**	**5**
1	0	0.2	0.1	0.3	0.3
2	0.0	0	0.4	0.2	0.3
3	0.0	0.5	0	0.0	0.4
4	0.3	0.2	0.2	0	0.1
5	0.1	0.4	0.2	0.1	0

The two search methods obtain the same optimal solution, (Q_1, Q_2, Q_3, Q_4, Q_5) = (41, 53, 56, 10, 0), and the optimal profit is $1,105.31. Table 6 compares their performance. The exhaustive search spend approximately 2.13 hours on examining around 29 million solutions. The number of solutions examined by the efficient search is less than half of that by the exhaustive search and the run time is only about 1.07 hours.

Table 6: Performance comparison in Example 2.

	Exhaustive search	**Efficient search**
Number of solutions examined	29,051,001	12,840,751
Run time (seconds)	7,652	3,847

5. NUMERICAL EXPERIMENT

It is quite common that managers are not aware of, or overlook demand substitutability. We are interest in the loss of not taking into account demand substitutability when allocating product quantities with limited storage space. Tables 7 and 8 contains data of Example 3 with five products and a storage capacity of 100. The sum of demand of the 5 products is 130.

Table 7: Parameter setting of Example 3.

Product	**Unit revenue**	**Unit cost**	**Demand**	**Unit salvage value**
1	25	7	20	0.7
2	20	5	40	0.5
3	15	6	20	0.6
4	10	3	10	0.3
5	10	5	40	0.5

Table 8: Substitute probabilities of Example 3.

	Product (j)				
Product (i)	**1**	**2**	**3**	**4**	**5**
1	0	0.2	0.2	0.3	0.3
2	0.1	0	0.1	0.5	0.2
3	0.0	0.1	0	0.3	0.1
4	0.1	0.1	0.2	0	0.4
5	0.1	0.1	0.1	0.2	0

If demand substitutability is not taken into account, managers would attempt to meet demand for products with greater unit profits. But the allocated quantity of a product does not exceed the demand of the product. The quantity allocation and the corresponding total profit are shown in Table 9. The efficient search method is applied to solve the same example with consideration of demand substitutability. The result is also provided in Table 8.

Table 9: Comparison in Example 3.

	Q_1	Q_2	Q_3	Q_4	Q_5	TP
Demand substitutability not considered	20	40	20	10	10	1260.0
Demand substitutability considered	24	44	25	1	6	1347.8

As can be seen, when demand substitutability is not considered, the allocated quantities of the first four products are equal to their respective demand as these products are more profitable. Only 10 units is allocated to product 5 due to the limit on storage space. When demand substitutability is taken into account, more units of products 1, 2 and 3 are allocated. The extra units of the three products are prepared in anticipation of that they will be purchased as substitute products by customers whose top choice is either product 4 or product 5. Since products 1, 2 and 3 have higher unit profit than products 4 and 5, it is favorable when they are purchased as substitute products. This example shows that total profit increases by $87.82 by taking advantage of demand substitutability and preparing extra units of more profitable products.

How many extra units of more profitable products to allocate depends on substitute probabilities. Table 10 breaks down the quantities of each products purchased by various types of customers. Among the 24 units of product 1, 20 of them are purchased by type 1 customers whose top choice is product 1. An expected quantity of 0.90 unit is purchased by type 4 customers as the substitute product and 2.96 units are bought by type 5 customers. Notice that there is an expected amount of 0.14 unit of unsold product 1. Preparing too many extra units may result in more ending inventory and more unsatisfied demand for less profitable products.

Table 10: Quantities purchased by various types of customers.

Product (i)	Q_i	**Customer**					E_i
		1	**2**	**3**	**4**	**5**	
1	24	20	0	0	0.90	2.96	0.14
2	44	0	40	0	0.90	2.96	0.14
3	25	0	0	20	1.80	3.20	0
4	1	0	0	0	1	0	0
5	6	0	0	0	0	6	0

Example 4 is utilized to further demonstrate how demand substitutability may effect quantity allocation and corresponding total profit. All the parameter values remain the same except demand for product 5, the least profitable product, is increased from 40 units to 60 units. The outcome is shown in Table 11.

Table 11: Comparison in Example 4.

	Q_1	Q_2	Q_3	Q_4	Q_5	**TP**
Demand substitutability not considered	20	40	20	10	10	1260.0
Demand substitutability considered	26	46	27	1	0	1407.4

The increase in the demand for product 5 does not change the quantity allocation if demand substitutability is not considered. However, when demand substitutability is taken into account, the 6 units allocated to product 5 in Example 3 are re-allocated to products 1, 2 and 3. When more type 5 customers are unable to purchase their top choice product, more customers will purchase higher profitable products as substitute. The change leads to greater total profit.

6. CONCLUSION

This research explores how demand substitutability affects quantity allocation in a single-period retail sales system with limited storage space. We utilize the constructed mathematical model to develop an efficient solution method that identify optimal quantity allocation to maximize expected total profit. Through the conducted numerical experiment, we demonstrate that when taking into account demand substitutability, the optimal quantity allocation changes and the optimal expected total profit is greater.

The proposed solution method may still not be efficient enough when solving problems of very large scale. Better solution methods for the considered problem can be developed in the future. It would be interesting to expand the problem to multiple periods. The problem with multiple storage sites is also challenging.

REFERENCES

Nagarajan, M., and Rajagopalan, S. (2008) Inventory models for substitutable products: optimal policies and heuristics, *Management Science*, **54**, 1453-1466.

Park, Y. B., and Yoon, S. J. (2011) A two-phase solution for the operation problem of vending machine systems with stock-out based, one-stage item substitution, *Proceedings of the 16th Annual International Conference on Industrial Engineering, Stuttgart, Germany*, 363-369.

Rajaram, K., and Tang, C. S. (2001) The impact of product substitution on retail merchandising, *European Journal of Operational Research*, **135**, 582-601.

Tayur, S., Ganeshan, R., and Magazine, M. (Eds.), (1999) Retail inventories and consumer choice in quantitative methods for supply chain management, Ch. 17.