An EOQ model with non-linear holding cost and partial backlogging under price and time dependent demand

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Abstract. In this work, a deterministic inventory model where demand depends on the selling price and time is developed. We assume that the holding cost is non-linear. Shortages are allowed and they are partially backlogged. So, the fraction of backlogged demand depends on the waiting time and on the stock-out period. The optimal policy is obtained by maximizing the total profit per unit time. We present a procedure to determine the economic lot size, the optimal inventory cycle and the maximum profit. The inventory system studied here admits as particular cases diverse EOQ models proposed in the literature. Finally, we provide some numerical examples to illustrate the theoretical results previously exposed.

Keywords: EOQ model, partial backlogging, price and time dependent demand, profit maximization

1. INTRODUCTION

As is well known, the inventory theory develops the necessary methodology to determine the optimal decisions on when a replenishment order should be made and how much should be ordered. This involves coordinating the functions of purchasing, manufacturing and distribution, which are made in business organizations in order to maximize its profit. In the organizational structure of enterprises, there are several departments involved in inventory management, from planning and marketing departments to the sales department, through the departments of finance, purchasing, production, distribution, quality and customer support. For this reason, the stocks have a high impact on the economic performance of the companies and, hence, the adequate management of the inventories is very important in the business organizations.

This work analyzes an inventory system of an item whose price is fixed by the market and, therefore, it is exogenous to the firm. Demand depends on time and is also price-sensitive. So, we suppose that demand is ramp-type and it additively combines the effects of a selling price function and a time function. The ramp-type demand has been used by other authors (see, for example, Karmakar and Choudhury, 2014; Kumar and Rajput, 2015, 2016; Manna et al. 2016; Skouri et al., 2009, 2011; and Yadav et al. 2016). More specifically, we suppose that when the net stock is positive, the part of demand which depends on time is a power function (i.e., it has a power demand pattern using the terminology of Naddor, 1966), while in the stock-out period, demand is independent of time. Since Naddor introduced the power demand pattern, several authors have developed inventory models with this hypothesis about demand (see, for example, Mandal and Islam, 2013; Palanivel and Uthayakumar, 2014; Rajeswari et al., 2015; Sicilia et al. 2012a, 2013; and Singh and Sehgal, 2011).

Also, we assume that shortages are allowed, but only a variable fraction of demand during a stock-out period is backlogged. This behavior face to shortage have also assumed by other authors. Among other works about variable partial backlogging, we can cite the papers by Pando et al. (2013, 2014), San-José et al. (2005, 2007), and Sicilia et al. (2009, 2012b).

Moreover, it is also considered that the holding cost is a non-linear function of time. Some recent papers on inventory systems with non-linear holding cost are the following: Pando et al. (2012), San-José et al. (2015), Sazvar et al. (2013), and Tripathi et al. 2016.

The organization of the paper is as follows. Section 2 describes the hypothesis on which the inventory system model is based and shows the notation used throughout the paper. Then, the inventory problem is mathematically formulated to maximize the total profit per unit time, by considering the purchasing cost, the ordering cost, the holding cost, the shortage cost and the selling price. Section 4 presents several results to characterize the optimal inventory policies as function of the input parameters of the system. Section 5 gives some numerical examples to illustrate the theoretical results previously provided. Finally, Section 6 concludes this study, summarizing the contributions of the new inventory model here studied and presenting possible directions for future research.

2. HYPOTHESIS AND NOTATION

The inventory system is based on the following assumptions. In the inventory, there is a single item with independent demand. The planning horizon is infinite and the replenishment is instantaneous. The cost of placing an order is known and independent of the quantity ordered. The purchasing cost and the selling price are known and constant. The unit holding cost is a potential function of time in storage. The inventory is continuously revised. The model allows shortages, which are partially backlogged and remaining are lost sales. The fraction of backlogged demand is a bivariate function of time the customer waits before receiving the items and on time lapsed since the break in the stock took place. The unit backorder cost includes a fixed cost and a cost proportional to the time for which the unit remains backordered. The goodwill cost of a lost sale is a known constant. The demand pattern is price- and ramp-type time-dependent. So, we suppose that in the stock-in period, demand is the sum of a price-dependent function and a power-time function, while in the stock-out period, demand only depends on price.

We will use the notation shown in Table 1 throughout this paper.

Table 1: Notations.

K	ordering cost (> 0)
p	unit purchasing cost (> 0)
s	unit selling price ($s > p$)
$H(t)$	holding cost per unit held in stock during t units
δ	elasticity of the holding cost (≥ 1)
ω_o	fixed cost per backordered unit
ω	backorder cost per unit and per unit time
π_o	goodwill cost of a lost sale
$D(t)$	demand rate at time t
$\alpha(s)$	part of demand that depends on the selling price
γ_1	average demand depends on time during the stock-in period
γ_2	average demand depends on time during the stock-out period
n	demand pattern index
$I(t)$	inventory level at time t
τ	stock-in period
Ψ	stock-out period
T	inventory cycle
$\beta(y,\Psi)$	fraction of backlogged demand
ρ	maximum fraction of backlogged demand
a	range of fraction of backlogged demand
Q	lot size per cycle
$B(\tau,\Psi)$	profit per unit time
$C(\tau,\Psi)$	inventory cost per unit time
$\sigma(s)$	auxiliary parameter, $\sigma(s) = h[\alpha(s)+(\delta+1)/(n\delta+1)\,\gamma_1]$
μ_1	auxiliary parameter, $\mu_1 = (\gamma_1-\gamma_2)(s-p) + (\alpha(s)+\gamma_2)[(\pi_o+s-p)(1-\rho+a/2) + \omega_o(\rho-a/2)$
μ_2	auxiliary parameter, $\mu_2 = (\alpha(s)+\gamma_2)\omega(3\rho-2a)/6$
Λ	auxiliary parameter, $\Lambda = \delta(\mu_1)^{1+1/\delta}-(\delta+1)\sigma(s)^{1/\delta}$

3. THE MATHEMATICAL MODEL

Based on the above assumptions about demand, the demand rate can be expressed as

$$D(t)=\begin{cases}\alpha(s)+\frac{\gamma_1}{n}\left(\frac{t}{\tau}\right)^{\frac{1}{n}-1} & \text{if } 0\le t<\tau\\ \alpha(s)+\gamma_2 & \text{if } \tau\le t<T\end{cases} \tag{1}$$

where $\alpha(s)$ represents the part of demand that depends on the selling price and γ_1 and γ_2, with $\gamma_1 \geq \gamma_2 > 0$, are the average demand depend on time during the stock-in and stock-out periods, respectively. We suppose as in Soni (2013) and Wu et al. (2014) that $\alpha(s) > 0$.

In this paper, we consider that the fraction of backlogged demand, which represents the behavior of the customers face to the shortage, is the function dependent of the waiting time and the stock-out period given by

$$\beta(y,\Psi) = \begin{cases} \rho - a\frac{y}{\Psi} & \text{if } 0 \le y \le \Psi \\ 0 & \text{otherwise} \end{cases} \qquad (2)$$

where ρ and a denote the maximum fraction of backlogged demand and the range of this fraction, respectively. Evidently, we suppose that $0 \le a \le \rho \le 1$.

So, the net inventory level is governed by the differential equation

$$\frac{dI(t)}{dt} = \begin{cases} -D(t) & \text{if } 0 < t < \tau \\ -D(t)\left(\rho - a\frac{T-t}{\Psi}\right) & \text{if } \tau < t < T \end{cases} \qquad (3)$$

with the initial condition $I(\tau) = 0$. Solving this equation, we obtain

$$I(t) = \begin{cases} \alpha(s)(\tau - t) + \gamma_1\tau\left[1 - \left(\frac{t}{\tau}\right)^{1/n}\right] & \text{if } 0 \le t \le \tau \\ \left(\alpha(s) + \gamma_2\right)(\tau - t)\left(\rho - a + a\frac{t-\tau}{2\Psi}\right) & \text{if } \tau < t < T \end{cases} \qquad (4)$$

Thus, the maximum stock level is $I(0) = (\alpha(s) + \gamma_1)\tau$ and the minimum net stock level is $I(T) = (\alpha(s) + \gamma_2)(a/2-\rho)\Psi$. Consequently, the lot size is $Q = I(0)-I(T)$. After a few algebraic manipulations, the lot size per cycle can be rewritten as

$$Q = \left(\alpha(s) + \gamma_1\right)T - \left(\alpha(s) + \gamma_2\right)\left(1 - \rho + \frac{a}{2}\right)\Psi - \left(\gamma_1 - \gamma_2\right)\Psi \qquad (5)$$

Taking into account the above assumptions, the total profit per cycle $G(\tau,\Psi)$ can be obtained as the difference between the revenue per cycle and the sum of the ordering cost, the purchasing cost, the holding cost, the backordering cost and the lost sale cost per cycle. It is evident that the revenue per cycle is sQ, the ordering cost is K, the purchasing cost is pQ and the holding cost is given by

$$HC(\tau) = \int_0^\tau H(t)[-I'(t)]dt \qquad (6)$$

where $H(t) = ht^\delta$, with $\delta \ge 1$. From (4), we obtain

$$HC(\tau) = \frac{\sigma(s)}{\delta+1}\tau^{\delta+1}, \qquad (7)$$

being

$$\sigma(s) = h\left[\alpha(s) + (\delta+1)/(n\delta+1)\gamma_1\right]. \qquad (8)$$

The backordering cost is given by

$$BC(\Psi) = -\omega_o I(T) + \omega\int_\tau^T [-I(t)]dt \qquad (9)$$

And, using again (4), we have

$$BC(\Psi) = \left(\alpha(s) + \gamma_2\right)\Psi\left[\omega_o\left(\rho - \frac{a}{2}\right) + \omega\frac{3\rho-2a}{6}\Psi\right] \qquad (10)$$

Finally, the goodwill cost per cycle is $\pi_o L(\Psi)$, where

$$L(\Psi) = \left(\alpha(s) + \gamma_2\right)\left(1 - \rho + \frac{a}{2}\right)\Psi \qquad (11)$$

Therefore, the total profit per cycle is given by

$$G(\tau,\Psi) = (s-p)Q - K - HC(\tau) - BC(\Psi) - \pi_o L(\Psi) \qquad (12)$$

From (5), it follows that the profit per unit time is

$$B(\tau,\Psi) = \frac{G(\tau,\Psi)}{\tau+\Psi} = (s-p)\left(\alpha(s) + \gamma_1\right) - C(\tau,\Psi), \qquad (13)$$

where

$$C(\tau,\Psi) = \frac{K + HC(\tau) + BC(\Psi) + (\pi_o + s - p)L(\Psi) + (\gamma_1 - \gamma_2)(s-p)\Psi}{\tau+\Psi} \qquad (14)$$

Thus, the optimization problem addressed in this paper consists of finding the values of the decision variables that minimizing the function $C(\tau,\Psi)$ given by (14).

4. ANALYSIS OF THE PROBLEM

Firstly, we will consider a fixed value for the variable Ψ. Therefore, let us define the function $C_\Psi(\tau)=C(\tau,\Psi)$. An interesting property of this function is the given by the following result.

Lemma 1. *The function $C_\Psi(\tau)$ is strictly convex and it attains its minimum at a non-negative point τ_Ψ^* that is a zero of the function $f_\Psi(\tau)$ given by*

$$f_\Psi(\tau) = (\delta+1)\sigma(s)\tau^\delta - \frac{\sigma(s)\tau^{\delta+1} + (\delta+1)\left(K + \mu_1\Psi + \mu_2\Psi^2\right)}{\tau+\Psi}, \qquad (15)$$

where

$$\mu_1 = (\alpha(s) + \gamma_2)\left[(\pi_o + s - p - \omega_o)\left(\frac{a}{2} - \rho\right) + \pi_o + s - p\right] + (\gamma_1 - \gamma_2)(s-p) \qquad (16)$$

$$\mu_2 = (\alpha(s) + \gamma_2)\omega\frac{3\rho-2a}{6} \qquad (17)$$

Proof. Indeed, from (14), (16) and (17), it follows that

$$\frac{dC_\Psi(\tau)}{d\tau} = \frac{\sigma(s)\tau^\delta(\tau+\Psi) - \left[K + \sigma(s)/(\delta+1)\tau^{\delta+1} + \mu_1\Psi + \mu_2\Psi^2\right]}{(\tau+\Psi)^2} \qquad (18)$$

and

$$\frac{d^2C_\Psi(\tau)}{d\tau^2} = \frac{\frac{\delta-1}{\delta+1}\sigma(s)\left(\delta\tau + 2(\delta+1)\Psi\right)\tau^\delta + \delta\sigma(s)\Psi^2\tau^{\delta-1} + 2(K + \mu_1\Psi + \mu_2\Psi^2)}{(\tau+\Psi)^3} \qquad (19)$$

So, we see that $d^2C_\Psi(\tau)/d\tau^2>0$, which proves the strict convexity of the function $C_\Psi(\tau)$. Moreover, from (14), it follows that $\lim_{\tau\to\infty} C_\Psi(\tau) = \infty$, and by (18), it may be concluded that

$$\left.\frac{dC_\Psi(\tau)}{d\tau}\right|_{\tau=0} = -\frac{K+\mu_1\Psi+\mu_2\Psi^2}{\Psi} < 0.$$

From this, we deduce that the minimum value of $C_\Psi(\tau)$, denoted by τ^*_Ψ, is obtained resolving the equation $dC_\Psi(\tau)/d\tau = 0$, or equivalently, $f_\Psi(\tau) = 0$, because $f_\Psi(\tau) = (\delta+1)(\tau+\Psi)dC_\Psi(\tau)/d\tau$.

Note that, in general, we have not an expression in closed-form for the value τ^*_Ψ.

Taking into account Lemma 1, we can now say that our problem consists of determining the variable Ψ that minimizes the univariate function $C(\tau^*_\Psi,\Psi) = \sigma(s)(\tau^*_\Psi)^\delta$.

For this reason, we shall show some interesting properties of that function.

Lemma 2. *The function $C(\tau^*_\Psi,\Psi)$ is of class C^1 and*

$$sign\left(\frac{dC(\tau^*_\Psi,\Psi)}{d\Psi}\right) = sign(g(\Psi)), \tag{20}$$

where

$$g(\Psi) = \delta(\mu_1+2\mu_2\Psi)^{1+1/\delta} + (\delta+1)\sigma(s)^{1/\delta}(\mu_2\Psi^2 - K) \tag{21}$$

Moreover,

$$sign\left(g'(\Psi)\right) = sign(\mu_2). \tag{22}$$

Proof. Firstly, it is clear that $C(\tau^*_\Psi,\Psi)$ is a continuous function. Now, taking into account that $f_\Psi(\tau^*_\Psi) = 0$, we obtain

$$\frac{dC(\tau^*_\Psi,\Psi)}{d\Psi} = \frac{(\mu_1+2\mu_2\Psi)(\tau^*_\Psi+\Psi) - \left[K+\sigma(s)/(\delta+1)(\tau^*_\Psi)^{\delta+1}+\mu_1\Psi+\mu_2\Psi^2\right]}{(\tau^*_\Psi+\Psi)^2} \tag{23}$$

and, from (14), it follows that

$$\frac{dC(\tau^*_\Psi,\Psi)}{d\Psi} = \frac{\mu_1+2\mu_2\Psi - C(\tau^*_\Psi,\Psi)}{\tau^*_\Psi+\Psi} \tag{24}$$

Hence, $sign(dC(\tau^*_\Psi,\Psi)/d\Psi) = sign(\mu_1+\mu_2\Psi - C(\tau^*_\Psi,\Psi))$.

After a few algebraic manipulations, from (21), we obtain

$$\begin{aligned} g(\Psi) &= \delta(\mu_1+2\mu_2\Psi)^{1+1/\delta} + (\delta+1)\sigma(s)^{1/\delta}\Psi(\mu_1+2\mu_2\Psi) \\ &+ \sigma(s)^{1+1/\delta}(\tau^*_\Psi)^{\delta+1} - (\delta+1)\sigma(s)^{1/\delta}(\tau^*_\Psi+\Psi)C(\tau^*_\Psi,\Psi) \end{aligned}$$

Since $f_\Psi(\tau^*_\Psi) = 0$, we can rewritten $g(\Psi)$ as

$$\begin{aligned} g(\Psi) &= \delta\left[(\mu_1+2\mu_2\Psi)^{1+1/\delta} - C(\tau^*_\Psi,\Psi)^{1+1/\delta}\right] \\ &+ (\delta+1)\sigma(s)^{1/\delta}\Psi\left[\mu_1+2\mu_2\Psi - C(\tau^*_\Psi,\Psi)\right] \end{aligned} \tag{25}$$

This proves (20).

Deriving in Eq. (21), we see that

$$g'(\Psi) = 2(1+\delta)\mu_2\left[(\mu_1+2\mu_2\Psi)^{1/\delta} + \sigma(s)^{1/\delta}\Psi\right], \tag{26}$$

which establishes the formula (22).

According to the above lemma, we need first to find Ψ^* (the optimum value of Ψ) by analyzing the behavior of the function $g(\Psi)$. Then, we must solve the equation $f_{\Psi^*}(\tau)= 0$ to obtain the optimum value of τ.

Therefore, let us first examine the function $g(\Psi)$ to determine the optimum value of Ψ.

Theorem 1. *Let* $\Lambda = \delta\mu_1^{1+1/\delta} - (\delta+1)K\sigma(s)^{1/\delta}$, and *$\sigma(s)$, μ_1, μ_2 and $g(\Psi)$ be given, respectively by (8), (16), (17) and (21). The optimal stock-out period can be obtained as follows:*

(a) If $\Lambda<0$ and $\mu_2=0$, then the function $C(\tau^*_\Psi,\Psi)$ attains its minimum at the point $\Psi^*=\infty$.

(b) If $\Lambda<0$ and $\mu_2>0$, then the function $C(\tau^*_\Psi,\Psi)$ attains its minimum at the point $\Psi^*=arg_{\Psi>0}\{g(\Psi)=0\}$.

(c) If $\Lambda=0$ and $\mu_2=0$, then the function $C(\tau^*_\Psi,\Psi)$ attains its minimum at all points Ψ of the interval $[0, \infty)$.

(d) If $\Lambda=0$ and $\mu_2>0$, then the function $C(\tau^*_\Psi,\Psi)$ attains its minimum at the point $\Psi^*=0$.

(e) If $\Lambda>0$, then the function $C(\tau^*_\Psi,\Psi)$ attains its minimum at the point $\Psi^*=0$.

Proof. By (22), we consider the following two cases:

I. Case $\mu_2=0$.
   We have that $g(\Psi)$ is a constant function with value Λ. Hence, three cases can occur:
   (A) If $\Lambda<0$, then the function $C(\tau^*_\Psi,\Psi)$ is strictly decreasing and, therefore, it attains its minimum value at $\Psi^* = \infty$.
   (B) If $\Lambda=0$, then the function $C(\tau^*_\Psi,\Psi)$ is constant on the interval $[0, \infty)$. Thus, it attains its minimum value at all points of such interval.
   (C) If $\Lambda>0$, then the function $C(\tau^*_\Psi,\Psi)$ is strictly increasing on the interval $[0, \infty)$ and, therefore, it attains its minimum value at $\Psi^* = 0$.

II. Case $\mu_2>0$. By Lemma 2, we know that now the function $g(\Psi)$ is strictly increasing for all $\Psi > 0$. Taking into account that $g(0) = \Lambda$ and $\lim_{\Psi\to\infty} g(\Psi) = \infty$, we can consider the following cases:
   (A) If $\Lambda<0$, then the function $g(\Psi)$ has a unique positive root Ψ_1. Thus, the function $C(\tau^*_\Psi,\Psi)$ is strictly decreasing on the interval $(0, \Psi_1)$ and it is strictly increasing on the interval (Ψ_1, ∞). Therefore, $C(\tau^*_\Psi,\Psi)$ attains its minimum at the point $\Psi^* = \Psi_1$.
   (B) If $\Lambda\geq 0$, then the function $C(\tau^*_\Psi,\Psi)$ is strictly increasing for all $\Psi > 0$ and, hence, it attains its minimum value at $\Psi^* = 0$.

Next, we will study the inventory system when the optimal stock-out period is zero.

Corollary 1. *If the optimal stock-out period is $\Psi^* = 0$, then:*
(1) The optimal inventory cycle is

$$\tau^* = \tau_o^* = \left(\frac{(\delta+1)K}{\delta\sigma(s)}\right)^{1/(\delta+1)} \quad (27)$$

(2) The optimal order quantity is

$$Q^* = Q_o = (\alpha(s)+\gamma_1)\tau_o^* \quad (28)$$

(3) The maximum profit per unit time is

$$B^* = B_o^* = B(\tau^*, \Psi^*) = (s-p)(\alpha(s)+\gamma_1) - \sigma(s)(\tau_o^*)^{\delta} \quad (29)$$

Proof. By Lemma 1, we have that τ_o^* is the zero of the function

$$f_o(\tau) = \delta\sigma(s)\tau^{\delta} - \frac{(\delta+1)K}{\tau} \quad (30)$$

This gives (27). The rest of the proof is immediate by (5), (13) and (14).

Next, we will show some properties of the inventory system when the optimal stock-out is not finite.

Corollary 2. *Suppose that $\Lambda < 0$ and $\mu_2 = 0$. Then:*
(1) The optimal stock-in period is

$$\tau^* = \tau_\infty^* = \left(\frac{\mu_1}{\sigma(s)}\right)^{1/\delta} \quad (31)$$

(2) The maximum profit per unit time is

$$B^* = B_\infty^* = (s-p)(\alpha(s)+\gamma_1) - \mu_1 \quad (32)$$

Proof. Applying Theorem 1, we obtain $\Psi^* = \infty$. By Lemma 1, we deduce that now the optimal stock-in period can be obtained by resolving the equation $f_\infty(\tau) = 0$, where

$$f_\infty(\tau) = \lim_{\Psi\to\infty} f_\Psi(\tau) = (\delta+1)\left(\sigma(s)\tau^{\delta} - \mu_1\right), \quad (33)$$

which yields Eq. (31).

Since $C(\tau_\Psi^*, \Psi) = \sigma(s)(\tau_\Psi^*)^{\delta}$ for all Ψ, we deduce that, in this hypothetical situation, the profit per unit time should be the expression given by (32).

Finally, we will analyze the inventory system when $\Lambda\mu_2 < 0$.

Corollary 3. *If the optimal stock-out period is obtained as $\Psi^* = \arg_{\Psi>0}\{g(\Psi)=0\}$, then:*

(1) The optimal stock-in cycle is

$$\tau^* = \left(\frac{\mu_1 + 2\mu_2\Psi^*}{\sigma(s)}\right)^{1/\delta} \quad (34)$$

(2) The optimal order quantity is

$$Q^* = \left(\alpha(s)+\gamma_1\right)\left(\frac{\mu_1+2\mu_2\Psi^*}{\sigma(s)}\right)^{1/\delta} + \left(\alpha(s)+\gamma_2\right)\left(\rho - \frac{a}{2}\right)\Psi^* \quad (35)$$

(3) The maximum profit per unit time is

$$B^* = (s-p)(\alpha(s)+\gamma_1) - \left(\mu_1 + 2\mu_2\Psi^*\right) \quad (36)$$

Proof. It follows easily from Lemmas 1 and 2 and Theorem 1, taking into account that, in this case, the following equations are verified:

$$\begin{aligned} C(\tau^*, \Psi^*) &= \sigma(s)(\tau^*)^{\delta} \\ C(\tau^*, \Psi^*) &= \mu_1 + 2\mu_2\Psi^* \end{aligned} \quad (37)$$

4.1 Particular models

The inventory system presented in this paper includes as special cases different inventory models, some of which have been studied before by other authors. Thus, we can obtain as special cases of the proposed system the following models:

4.1.1 An inventory model with power demand, partial backlogging and linear holding cost

If we consider that the holding cost per unit and per unit time is constant, that is, $\delta = 1$, then we can find an expression in closed-form for the optimal inventory policy in all the possible cases.

We first calculate the optimal stock-out period. By Theorem 1, we only need consider the case when $\Lambda < 0$ and $\mu_2 > 0$.

From (21), we see that in such situation, $\Psi^* = \Psi_1$, where

$$\Psi_1 = \arg_{\Psi>0}\{g(\Psi)=0\} = \frac{-\mu_1 + \sqrt{\mu_1^2 + \left(K\sigma(s) - \mu_1^2/2\right)\left(2+\sigma(s)/\mu_2\right)}}{\sigma(s)+2\mu_2} \quad (38)$$

Next, we determine the optimal stock-in period. From (15), we see that

$$\tau_\Psi^* = -\Psi + \sqrt{\Psi^2 + \frac{2}{\sigma(s)}\left(K + \mu_1\Psi + \mu_2\Psi^2\right)}, \quad (39)$$

which is sufficient to show our assertion.

4.1.2 An inventory model with power demand, non-linear holding cost and fixed partial backlogging

If we consider that the range of variation for the fraction of backlogged demand is zero, then we obtain $\beta(y,\Psi) = \rho$ for $0 \le y \le \Psi$. So, the fraction of backlogged demand does not depend neither on the waiting time for the next replenishment nor on the length of the stock-out period.

Table 2: Optimal policies when $a = 0$

	$\rho\omega = 0$	$\rho\omega > 0$
$\delta\theta^{1+1/\delta}<(\delta+1)\sigma(s)^{1/\delta}$	$([\theta/\sigma(s)]^{1/\delta},\infty)$	$(\tau^*_{\tilde{\Psi}},\tilde{\Psi})$
$\delta\theta^{1+1/\delta}=(\delta+1)\sigma(s)^{1/\delta}$	all point (τ^*_{Ψ},Ψ)	$(\tau^*_o,0)$
$\delta\theta^{1+1/\delta}>(\delta+1)\sigma(s)^{1/\delta}$	$(\tau^*_o,0)$	$(\tau^*_o,0)$

where τ^*_o is given by (27)

$\tilde{\Psi} = \arg_{\Psi>0}\{g(\Psi)=0\}$

Since, $sign(\mu_2)=sign(\rho\omega)$ and μ_1 is reduced to θ, where

$$\theta = (\alpha(s)+\gamma_2)[\rho\omega_o + (1-\rho)(\pi_o + s - p)] + (\gamma_1 - \gamma_2)(s-p), \tag{40}$$

the optimal inventory policy can be calculated as is shown in Table 2.

4.1.3 The inventory model developed by San-José et al. (2007)

If we consider that $n = 1$, $\delta = 1$, $\alpha(s)\to 0^+$ and $\gamma_1 = \gamma_2$, we obtain the model with constant demand and variable fraction of backlogging given by (2). This inventory models was studied by San-José et al. (2007).

We leave it to reader to verify that the optimal policy there shown is the same as the obtained one by through Theorem 1 of this paper.

4.1.4 The inventory model with power demand pattern without shortage

If we suppose that $\rho\omega_o + (1-\rho)\pi_o\to\infty$, then $\mu_1\to\infty$ and, applying Theorem 1, it follows that the optimal stock-out period is zero. From Corollary 1, we see that the optimal inventory cycle is τ^*_o, which is the optimal policy for the inventory model with power demand in which shortage is not allowed.

If in addition to $\rho\omega_o + (1-\rho)\pi_o\to\infty$, we assume that $\delta = 1$, $\alpha(s)\to 0^+$, it results the inventory model with power demand pattern and without shortage studied by Sicilia et al. (2012a), using the notation $r = \gamma_1$ and $A = K$.

5. NUMERICAL EXAMPLES

Next, we present several numerical examples to illustrate the previous results.

Example 1. Let us consider an inventory system with the assumptions given in this paper in which the input parameters are the following: $\alpha(s)$=70-5s, $\gamma_1 = \gamma_2$ =15, n = 1, K = 250, h = 0.53, ω_o = 0.2, ω = 2, π_o = 10, p = 8, s = 12, ρ= 0.9, a = 0.9 and δ = 1. We first compute μ_1 = 194.75 and μ_2 = 7.5. Then, we obtain Λ = 31302.6. Applying Theorem 1, we conclude that the optimal stock-out period is Ψ^* = 0. Hence, by Corollary 1, it follows that the optimal inventory cycle is $T^* = \tau^*_o = 6.14295$ and the optimal profit is B^*= 18.6059.

Example 2. We suppose the same parameters as in Example 1, but change the value of a to a = 0.1. Now, we have μ_1 = 56.75, μ_2 = 20.8333 and Λ = -3404.44. From Theorem 1, we see that the optimal stock-out period is Ψ^* = $\arg_{\Psi>0}\{g(\Psi)=0\}$ = 0.565275. Note that, as δ = 1, the value of Ψ^* can be also calculated using (38). From (34), the optimal stock-in period is τ^* = 6.06061, hence the optimal lot size is Q^*= 157.168 and the optimal profit is B^*= 19.6969.

Example 3. We suppose the same parameters as in Example 2, but modify the value of ω to ω = 0. We have μ_2= 0, while the values of μ_1 and Λ remain unalterable. Applying Theorem 1, we find that the optimal stock-out period is infinite, that is, the inventory system does not exist in proper sense.

6. CONCLUSIONS

In this paper, an inventory model for a single item, where the demand pattern is price- and ramp-type time-dependent is developed. We suppose that the price is fixed by the market. The inventory system allows shortages, which are partially backlogged according to a bivariate function, which depends on both waiting time and stock-out period. Also, we consider that the holding cost is a potential function of time in stock. We have presented an exact solution procedure for the inventory system. In addition, if the holding cost per unit is a linear function of time, then we have given an expression in closed-form for the optimal policy. Moreover, the proposed inventory system includes several inventory models of the literature on the topic.

Some directions for further research may be the following: (i) assume a finite rate of replenishment, (ii) consider other function to model the partial backlogging, (iii) suppose an item deteriorates over time, (iv) assume discounts in purchasing costs and (v) consider other functions for the unit holding cost.

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