Statistical Mechanics Model for M/M/s Queueing System with Balking

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Abstract. Behavior that a customer who has arrived at a queueing system leaves without joining the queue is known as the phenomenon of balking. Then, the phenomenon of balking has been treated as one of stochastic factors in the equilibrium analysis of queueing systems. Therefore, queueing systems with balking have been studied continually as one of significant subjects. Recently, the statistical mechanics model to analyze the equilibrium state of the M/M/1 queueing system with balking has been established based on the concept of the statistical mechanics explained by the relationship of the entropy and potential energy. In this study, the novel statistical mechanics model to analyze the equilibrium state of the M/M/s queueing system with balking is constructed by expanding the statistical mechanics model to analyze the equilibrium state of the M/M/1 queueing system with balking. By using the statistical mechanics model constructed in this study, we can derive the equilibrium state probabilities and arrival rates of the M/M/s queueing system with balking depending on each queue length. Furthermore, it is shown through mathematical verifications that the novel statistical mechanics model is applicable to various types of Markovian queueing systems. We conclude that the novel statistical mechanics model in this study have general versatility to analyze Markovian queueing systems.

Keywords: Balking, Statistical Mechanics, Entropy, Potential Energy, M/M/s Queueing System

1. INTRODUCTION

Queueing theory is well known as a theory concerned with stochastic fluctuations in the congestion phenomenon of the system. Many results of research about queueing theory have been presented. As one of those results, Guiasu (1986) has suggested a queueing model based on the maximum entropy principle as a mathematical model for analyzing the equilibrium state of the queueing system. Moreover, based on the maximum entropy principle, Arizono et al. (1991) has proposed an entropy model for analyzing the equilibrium state probability distribution of the M/M/s queueing system. Additionally, there are some studies that associated the concept of entropy with various queueing systems. See Wang et al. (2002), Jain and Dhakad (2003), Prabhakar (2003), Borzadaran (2009) and Singh and Tiwari (2013).

Meanwhile, the phenomenon of balking has been treated as one of stochastic factors in the equilibrium analysis of queueing systems. Then, the balking means the behavior that a customer who has arrived at a queueing system leaves without joining the queue. Under consideration of the balking phenomenon, the arrival rate to the system is not constant, and varies depending on the queue length of the system. Queueing systems with balking are studied continually as one of significant subjects in the recent years. See Montazer-Haghighi et al. (1986), Abou-El-Ata and Hariri (1992), János (2011), El-Sherbiny (2012) and Jain et al. (2014). In the traditional equilibrium state analysis for queueing systems under the consideration of the balking phenomenon, it has been typically assumed that all arrival rates under the situation of each queue length are provided. For reference, Abou-El-Ata and Hariri (1992), János (2011) and El-Sherbiny (2012) have expressed arrival rates by a function depending on the queue length, respectively.

In recent year, a novel model based on the concept of the statistical mechanics for analyzing the equilibrium state of the M/M/1 queueing system with balking has been proposed by Tamura et al. (2015). The concept of the statistical mechanics is explained based on both the entropy and potential energy. The entropy is interpreted as the criterion for evaluating the microscopic irregularity of systems, and the potential energy is interpreted as the criterion for evaluating the instability of systems. In the statistical mechanics, the entropy maximum principle and the energy minimum principle are understood as indivisible concepts of the pair (Chandler 1987). Therefore, in the statistical mechanics, the equilibrium state of the system is described by the balance of the entropy and potential energy. Specifically, the equilibrium state probability distribution is explained based on the balance of the entropic force and the energy force defined by the change of state in the system.

In this study, we expand the statistical mechanics model presented by Tamura et al. (2015) for the M/M/1 queueing system. Then, the novel statistical mechanics model to analyze the equilibrium state of the M/M/s queueing system with balking is constructed. Further, the equilibrium state probability and arrival rate depending on each queue length are derived based on the statistical mechanics model constructed in this study. In addition, through some comparisons between some traditional results about Markovian queueing systems and the results by the statistical mechanics model for the M/M/s queueing system, the versatility of the constructed statistical mechanics model for the analysis of Markovian queueing models is confirmed.

2. STATISTICAL MECHANICS MODEL FOR M/M/s QUEUEING SYSTEM

The concept of the statistical mechanics is explained based on both the entropy and potential energy. The entropy is interpreted as the criterion for evaluating the microscopic irregularity of systems, and the potential energy is interpreted as a criterion for evaluating the instability of systems. Then, in the concept of the statistical mechanics, the equilibrium state probability distribution of a system is derived by the relationship of the entropy and potential energy. In this section, we address respective definitions of the entropy and potential energy in the M/M/s queueing system with balking first. Next, based on the definitions of the entropy and potential energy, the statistical mechanics model for analyzing the equilibrium state of the M/M/s queueing system with balking is discussed.

We consider the Markovian queueing system with s servers under consideration of the balking phenomenon. Then, each server has the service rate μ, where μ is known. And, it is assumed that the arrival rate in the condition without balking is known as λ. However, it is assumed that the arrival rate varies depending on the queue length under the consideration of the balking phenomenon. Then, we describe the arrival rate under the situation that the queue length is n as λ_n. By using this notation, the arrival rate λ_0 under the situation that the queue length is 0 is indicated as $\lambda_0 = \lambda$, because it is natural that the balking does not occur in the situation that there is an available vacant server. On the other hand, it is also natural to suppose the relation of $\lambda_n \geq \lambda_{n+1}$ for $n \geq 0$ as common sense. Finally, in the system design phase, as a guarantee that the state of the system does not diverge, we assume the traffic density ρ in the natural condition as $\rho = \lambda / (s\mu) < 1$ in this study.

In reference to Arizono et al. (1991) that have treated the entropy model for the M/M/s queueing system without balking, we define the statistical mechanics model for the M/M/s queueing system with balking. After this, we describe the state that the queue length is n as state n. Then, we define the state probability of state n as P_n. Additionally, the state probability vector $\vec{P}$ is presented as $\vec{P} = (P_0, P_1, P_2, \cdots, P_n, \cdots)$.

At first, whole states in the M/M/s queueing system are divided into two groups for $0 \leq n < s$ and $s \leq n$. Then, in the transition from state $n+1$ to state n in the group of $0 \leq n < s$, the total service rate varies depending on state n, that is, the total service rate in the transition from state $n+1$ to state n is given as $(n+1)\mu$. In such a case, state n of $0 \leq n < s$ can be divided into quasi-states (n,i), $1 \leq i \leq s!/n!$, consisting of $s!/n!$ cases by considering the number of cases in the transitions from state s to state n, $0 \leq n < s$. See Arizono et al. (1991). Thus, the state probability P_n in the case of $0 \leq n < s$ is represented as

$$P_n = \sum_{i=1}^{s!/n!} q_{n,i}, \ 0 \leq n < s, \tag{1}$$

where $q_{n,i}$ means the state probability of the quasi-state (n,i), $1 \leq i \leq s!/n!$.

On the other hand, in the group of $s \le n$, the service rate in the transition from state $n+1$ to state n is constant as $s\mu$. And, the transitions from state $n+1$ to state n occur if a service at either server of s servers is completed. Then, we divide state n in this group of $s \le n$ into quasi-states (n, j), $1 \le j \le s$, consisting of s cases by considering the number of cases in the transitions from state $n+1$ to state n. Thus, the state probability P_n in the case of $n \ge s$ is supposed as

$$P_n = \sum_{j=1}^{s} q_{n,j}, \; n \ge s. \tag{2}$$

where $q_{n,j}$ means the state probability of the quasi-state (n, j), $1 \le j \le s$.

In consequence, based on the quasi-state probabilities mentioned above, the entropy can be defined as

$$H\left(\vec{P}\right) = -\sum_{n=0}^{s-1}\sum_{i=1}^{s!/n!} q_{n,i} \ln q_{n,i} - \sum_{n=s}^{\infty}\sum_{j=1}^{s} q_{n,j} \ln q_{n,j}, \tag{3}$$

where interpret $\vec{P}$ as the state probability vector constituted by the state probabilities P_n based on Eqs.(1) and (2).

Next, we address the potential energy in the M/M/s queueing system with balking. As mentioned above, since the states of the M/M/s queueing system are divided into two groups of $0 \le n < s$ and $s \le n$, we should defined two kinds of the potential energy. Then, in the state of the group of $0 \le n < s$, waiting to receive the service does not occur because there are some available vacant servers. Therefore, we can consider that the balking does not occur in the case of $0 \le n < s$ even if a customer has arrived at the system increase. In such a case, Tamura et al. (2015) have indicated that the potential function can be defined as

$$g_1\left(n\right) = n, \; 0 \le n < s. \tag{4}$$

Then, the potential function $g_1\left(n\right)$ can be interpreted as the function indicating the potential based on the increase of busy servers. Note that, in this study, the value of the potential function is called simply the potential.

In contrast to the case of $0 \le n < s$, in the state of the group of $s \le n$, the customer who arrived newly to the M/M/s queueing system must wait by necessity. Then, separately from the potential based on the increase of busy servers, it is necessary to suppose the potential based on the increase of the customers waiting for service as the increase function against n in $s \le n$. In this case, the potential function based on the increase of the customers waiting for service as the increase function against n in $s \le n$ is described as $g_2\left(n\right)$. Then, we define $g_2\left(s\right) = 1$ as the unit amount of the potential based on the increase of customers waiting for service. In addition, we can suppose $g_2\left(n\right) = 0$ for $0 \le n < s$ in the potential function $g_2\left(n\right)$ based on the increase of the customer waiting for service, because there is no customer waiting for service and the balking does not occur in this situation.

In this study, based on the potential function $g_1\left(n\right)$, the potential energy $U_1\left(\vec{P}\right)$ due to the increase of busy servers can be defined as follows:

$$U_1\left(\vec{P}\right) = \sum_{n=0}^{s-1}\sum_{i=1}^{s!/n!} g_1\left(n\right) q_{n,i} + \sum_{n=s}^{\infty}\sum_{j=1}^{s} g_1\left(s-1\right) q_{n,j}. \tag{5}$$

Further, the potential energy $U_2\left(\vec{P}\right)$ due to the increase of customers waiting for service can be defined as

$$U_2\left(\vec{P}\right) = \sum_{n=s}^{\infty}\sum_{j=1}^{s} g_2\left(n\right) q_{n,j}. \tag{6}$$

In the concept of the statistical mechanics, the equilibrium state is described by the balance of the force due to the change in the state of the entropy and potential energy in the system. Here, the change of the state in the M/M/s queueing system can be shown by the change of the state probability P_n. Therefore, the entropic force and energy force associated with change of the state probability distribution of the M/M/s queueing system. It can be evaluated by the partial differential regarding the state probability vector $\vec{P}$ of each function $H\left(\vec{P}\right)$, $U_1\left(\vec{P}\right)$ and $U_2\left(\vec{P}\right)$. Furthermore, the directions of the entropic force and the energy force are opposite (See Tribus and McIrvine. 1971). In consequence, based on the concept of the statistical mechanics, the equilibrium state for the M/M/s queueing system is established on basis of the relationship of

$$\frac{\partial}{\partial \vec{P}} H\left(\vec{P}\right) = \omega_1 \frac{\partial}{\partial \vec{P}} U_1\left(\vec{P}\right) + \omega_2 \frac{\partial}{\partial \vec{P}} U_2\left(\vec{P}\right), \tag{7}$$

where ω_1 and ω_2 are given as undetermined multipliers for according the respective dimensions (units) of the energy forces associated with $U_1\left(\vec{P}\right)$ and $U_2\left(\vec{P}\right)$ to the dimension (unit) of the entropic force associated with $H\left(\vec{P}\right)$. In addition, because the queueing system is a probability system, we consider the following constraint:

$$\sum_{n=0}^{\infty} P_n = \sum_{n=0}^{s-1}\sum_{i=1}^{s!/n!} q_{n,i} + \sum_{n=s}^{\infty}\sum_{j=1}^{s} q_{n,j} = 1. \tag{8}$$

Thus, we define the statistical mechanics model for analyzing the equilibrium state of the M/M/s queueing system with balking as follows:

$$\begin{aligned}\Lambda\left(\vec{P}, \omega_1, \omega_2, \omega_3\right) = {} & \frac{\partial}{\partial \vec{P}} H\left(\vec{P}\right) \\ & - \omega_1 \frac{\partial}{\partial \vec{P}} U_1\left(\vec{P}\right) - \omega_2 \frac{\partial}{\partial \vec{P}} U_2\left(\vec{P}\right) \\ & - \omega_3 \frac{\partial}{\partial \vec{P}} \left(\sum_{n=0}^{s-1}\sum_{i=1}^{s!/n!} q_{n,i} + \sum_{n=s}^{\infty}\sum_{j=1}^{s} q_{n,j} - 1 \right) = \vec{0},\end{aligned} \tag{9}$$

where ω_3 indicate a undetermined multiplier for the constraint of Eq.(8). Each of undetermined multipliers can be understood as Lagrange multiplier. Then, the statistical mechanics model can be treated as a first-order condition regarding Lagrange function

3. DERIVATION OF EQUILIBRIUM STATE PROBABILITY DISTRIBUTION BASED ON STATISTICAL MECHANICS MODEL FOR M/M/S QUEUEING SYSTEM

From Eq.(9) presenting the statistical mechanics model, we obtain the following equation in the case of $0 \le n < s$:

$$\ln q_{n,i} = -\left(1+\omega_3\right) - \omega_1 g_1\left(n\right). \qquad (10)$$

Accordingly, the equilibrium state probability $q_{n,i}$ of the quasi-state (n,i), $0 \le n < s$ and $1 \le i \le s!/n!$, is derived as

$$q_{n,i} = e^{-(1+\omega_3)} e^{-\omega_1 g_1(n)}. \qquad (11)$$

Further, by representing $e^{-(1+\omega_3)}$ and $e^{-\omega_1}$ in Eq.(11) as α and β, the equilibrium state probability P_n of Eq.(1) can be obtained as:

$$P_n = \sum_{i=1}^{s!/n!} q_{n,i} = \frac{s!}{n!}\alpha\beta^{g_1(n)},\ 0 \le n < s. \qquad (12)$$

Likewise, in the case of $n \ge s$, we obtain

$$\ln q_{n,j} = -\left(1+\omega_3\right) - \omega_1 g_1\left(s-1\right) - \omega_2 g_2\left(n\right). \qquad (13)$$

Then, the equilibrium state probability $q_{n,j}$ of the quasi-state (n, j), $n \ge s$ and $1 \le j \le s$, is derived as

$$q_{n,j} = e^{-(1+\omega_3)} e^{-\omega_1 g_1(s-1)} e^{-\omega_2 g_2(n)}. \qquad (14)$$

Accordingly, by representing $e^{-\omega_2}$ in Eq.(14) as γ, the equilibrium state probability P_n of Eq.(2) can be obtained as follows:

$$P_n = \sum_{j=1}^{s} q_{n,j} = s\alpha\beta^{g_1(s-1)}\gamma^{g_2(n)},\ n \ge s. \qquad (15)$$

As the result, the equilibrium state probability distribution is described as:

$$P_n = \begin{cases} \dfrac{s!}{n!}\alpha\beta^{g_1(n)}, & 0 \le n < s, \\ s\alpha\beta^{g_1(s-1)}\gamma^{g_2(n)}, & n \ge s. \end{cases} \qquad (16)$$

On the other hand, in the traditional statistical equilibrium analysis, we have the system of equilibrium equations under the arrival rates λ_n and the service rate μ as follows:

$$\begin{cases} \lambda_0 P_0 = \mu P_1, & n = 0, \\ \lambda_{n-1}P_{n-1} + \left(n+1\right)\mu P_{n+1} = \left(\lambda_n + n\mu\right)P_n, & 1 \le n < s, \\ \lambda_{n-1}P_{n-1} + s\mu P_{n+1} = \left(\lambda_n + s\mu\right)P_n, & n \ge s. \end{cases} \qquad (17)$$

Then, as mentioned previously, waiting to receive the service does not occur because there are some available vacant servers in the states of $0 \le n < s$. Accordingly, we can consider that the balking does not occur in the case of $0 \le n < s$ even if customers increase. For this reason, the arrival rates λ_n, $0 \le n < s$, can be given as $\lambda_n = \lambda$. Thus, Eq.(17) is rewritten as:

$$\begin{cases} \lambda P_0 = \mu P_1, & n = 0, \\ \lambda P_{n-1} + \left(n+1\right)\mu P_{n+1} = \left(\lambda + n\mu\right)P_n, & 1 \le n < s, \\ \lambda_{n-1}P_{n-1} + s\mu P_{n+1} = \left(\lambda_n + s\mu\right)P_n, & n \ge s. \end{cases} \qquad (18)$$

Then based on Eq.(18), we can describe the equilibrium state probability distribution as:

$$P_n = \begin{cases} \dfrac{a^n}{n!}P_0, & 0 \le n < s, \\ \dfrac{a^{s-1}\prod_{k=s}^{n} a_{k-1}}{s!s^{n-s}}P_0, & n \ge s, \end{cases} \qquad (19)$$

where let $a_n = \lambda_n / \mu$, $n \ge s$.

On the other hand, by substituting $g_1\left(0\right) = 0$ and $g_1\left(1\right) = 1$ into Eq.(16), we have the followings:

$$\alpha = \frac{P_0}{s!}. \qquad (20)$$

$$P_1 = \beta P_0. \qquad (21)$$

Then, since we have the relation $\lambda P_0 = \mu P_1$ in Eq.(18), β in Eq.(21) can be obtained as

$$\beta = \frac{\lambda}{\mu} \equiv a, \qquad (22)$$

Furthermore, from Eq.(18), the following relationship can be shown:

$$\lambda P_{s-2} + s\mu P_s = \left\{\lambda + \left(s-1\right)\mu\right\}P_{s-1}. \qquad (23)$$

Hence, by substituting P_{s-1}, P_{s-2} and P_s of Eq.(16) into Eq.(23), γ can be derived as follows:

$$\gamma = \frac{\lambda}{s\mu} = \rho. \qquad (24)$$

Finally, based on the statistical mechanics model constructed in this study, the equilibrium state probability distribution of the M/M/s queueing system with balking is given as

$$P_n = \begin{cases} \dfrac{a^n}{n!} P_0, & 0 \le n < s, \\ \dfrac{a^s \rho^{g_2(n)-1}}{s!} P_0, & n \ge s, \end{cases} \tag{25}$$

where P_0 is represented as

$$P_0 = \left\{ \sum_{n=0}^{s-1} \frac{a^n}{n!} + \sum_{n=s}^{\infty} \frac{a^{s-1} \rho^{g_2(n)}}{(s-1)!} \right\}^{-1}. \tag{26}$$

Additionally, by comparing the Eq.(19) and Eq.(25), the arrival rate λ_n can be shown as follows:

$$\lambda_n = \begin{cases} \lambda, & 0 \le n < s, \\ \lambda \rho^{g_2(n+1)-g_2(n)-1}, & n \ge s. \end{cases} \tag{27}$$

Then, from the relationships of $\rho < 1$ as the prerequisite and $\lambda_n \ge \lambda_{n+1}$ as the assumption under the balking phenomenon, the following inequality for the potential function $g_2(n)$ is derived:

$$g_2(n+1) - g_2(n) \ge 1. \tag{28}$$

In this study, as a monotonically increasing function to satisfy the condition of Eq.(28), we assume the potential function $g_2(n)$ as

$$g_2(n) = (n - s + 1)^r, \tag{29}$$

where r means a system parameter to control the strength of balking, and then $r \ge 1$.

Under the potential function $g_2(n)$ of Eq.(29), the equilibrium state probability distribution is represented as

$$P_n = \begin{cases} \dfrac{a^n}{n!} P_0, & 0 \le n < s, \\ \dfrac{a^s}{s!} \rho^{(n-s+1)^r - 1} P_0, & n \ge s. \end{cases} \tag{30}$$

In this case of $r = 1.0$, the equilibrium state probability distribution can be easily obtained as

$$P_n = \begin{cases} \dfrac{a^n}{n!} P_0, & 0 \le n < s, \\ \dfrac{s^s}{s!} \rho^n P_0, & n \ge s. \end{cases} \tag{31}$$

Eq.(31) is known as the equilibrium state probability distribution without balking based on the traditional statistical equilibrium analysis in the M/M/s queueing system. From this result, it is obviously found that the equilibrium state probability distribution for the M/M/s queueing system without balking can be obtained as a special case of $r = 1$ in the statistical mechanics model constructed under the consideration of the balking phenomenon. Furthermore, as a matter of course, we can easily show that the statistical mechanics model for the M/M/1 queueing system with balking proposed by Tamura et al. (2015) is derived as a special case of $s = 1$ in the statistical mechanics model for the M/M/s queueing system with balking of this study. Then, the statistical mechanics model for the M/M/s queueing system in this study can be understood as the analysis model expanding the statistical mechanics model by Tamura et al. (2015). From the verification with regard to the M/M/s queueing system without balking and the M/M/1 queueing system with balking, we can conclude that the statistical mechanics model for the M/M/s queueing system under the consideration of the balking phenomenon is the versatile and convenient analysis model of Markovian queueing systems.

4. NUMERICAL ANALYSIS

In this section, we illustrate some numerical results based on the proposed statistical mechanics model for the M/M/s queueing system with balking. First, we set the service rate of each server as $\mu = 25.0$. Next, the traffic density in the condition without balking is given as $\rho = 0.8$. Under these settings, by changing the values of the system parameter r and the number of servers s, we investigate the influence of balking based on the proposed statistical mechanics model for the M/M/s queueing system with balking. As self-evidence, the arrival rate λ against each number of servers s is derived based on the relation of $\rho = \lambda / s\mu = 0.8$. As the result, the arrival rate λ in the condition without balking is different by the number of servers s. In the statistical mechanics model for the M/M/s queueing system with balking, the potential function $g_1(n)$ and $g_2(n)$ are respectively assumed as $g_1(n) = n$ and $g_2(n) = (n - s + 1)^r$ in this study.

In Figure 1, we illustrate the relationship between the queue length n and the arrival rate λ_n in the case of $s = 1$. From Figure 1, since the arrival rate λ_n satisfies the relationship of $\lambda_n \ge \lambda_{n+1}$, the influence of balking considered in this study has been successfully explained. In addition, it is found that the arrival rate λ_n decreases faster by a larger system parameter r. This fact means that the influence of balking becomes stronger when the system parameter r is large. On the other hand, remark that the situation of $r = 1.00$ corresponds to the M/M/1 queueing system without balking as mentioned previously. Therefore, the arrival rate λ_n are constant as $\lambda = 20$ in the case of $r = 1.00$. Moreover, we can see that the result in the case of $s = 1$ is consistent with the result of Tamura et al. (2015). As a consequence, we can confirm that the proposed statistical mechanics model for the M/M/s queueing system involves the statistical mechanics model for the M/M/1 queueing system presented by Tamura et al. (2015).

Similarly, in Figure 2 and 3, under some values of the system parameter r to control the strength of balking, we illustrate the relationships between the queue length n and the arrival rate λ_n in the case of $s=2$ and $s=3$, respectively. Since the traffic density ρ is fixed at 0.8, the arrival rate λ in the natural condition is derived as 40 in the case of $s=2$ and 60 in the case of $s=3$. Additionally, in the cases of $s=2$ and $s=3$, it can be seen that the phenomenon of balking occurs in $n \geq 2$ and $n \geq 3$, respectively. Conversely, the phenomenon of balking never occurs in $n < s$. Because a customer who has arrived at the system in the situation of $n < s$ can receive service immediately, this feature is rational. Just for the record, note that this feature has been shown in Eq.(27) mathematically.

Next, we illustrate the relationship between the queue length n and the equilibrium state probability P_n in the case of $s=1$, $s=2$ and $s=3$ in Figure 4, 5 and 6, respectively. According to the increase of the system parameter r, the shape of the equilibrium state probability distribution changes to sharper shapes. This feature also implies that the increase of the system parameter r makes the effect of balking stronger. That is, we can find that this feature has been derived from the result that the increase of r decreases λ_n.

Here, we can evaluate the expected arrival rate $\overline{\lambda_n}$ as

$$\overline{\lambda_n} = \sum_{n=0}^{\infty} \lambda_n P_n. \tag{32}$$

In Figure 7, we illustrate the relationship between the system parameter r and the expected arrival rate $\overline{\lambda_n}$ in the case of $s=3$. From Figure 7, it is seen that the expected arrival rate $\overline{\lambda_n}$ decreases according to the increase of the system parameter r. Therefore, we know that the influence of balking becomes stronger by the large system parameter r. Remark that the same feature has been confirmed in any case of the different number of servers.

5. CONCLUDING REMARKS

In this paper, based on the concept of the statistical mechanics, we have addressed the mathematical modeling for analyzing the M/M/s queueing system with balking. As the result, the novel statistical mechanics model has been successfully proposed for deriving the equilibrium state probability distribution. Therefore, it has been shown that the proposed statistical mechanics model can derive the rational equilibrium state probability distribution under the consideration of the balking phenomenon. Further, the

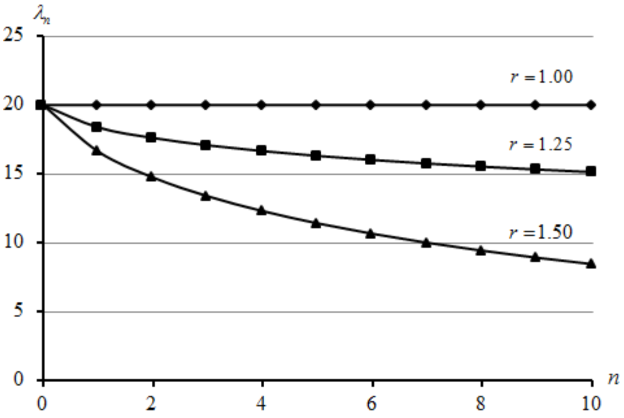

Figure 1: the relationship between the queue length n and the arrival rate λ_n in the case of $s=1$

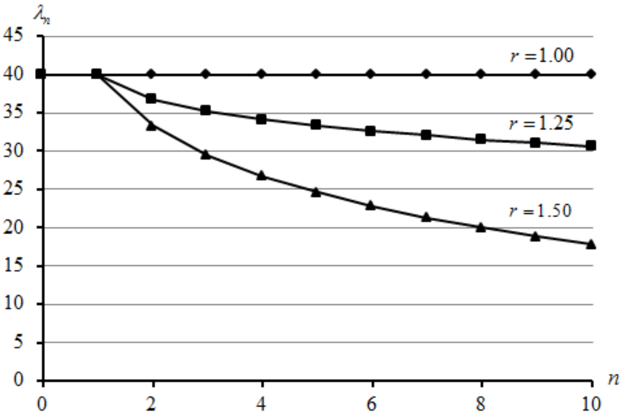

Figure 2: the relationship between the queue length n and the arrival rate λ_n in the case of $s=2$

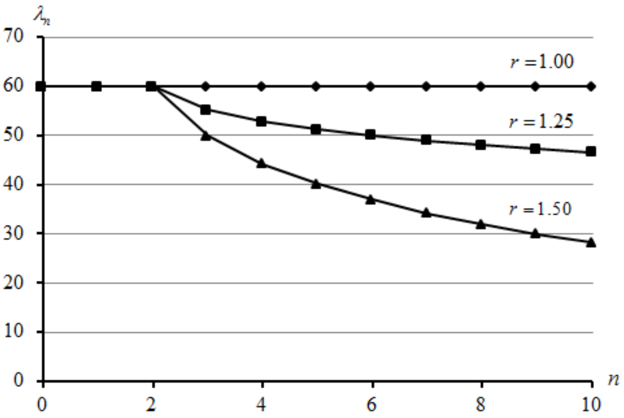

Figure 3: the relationship between the queue length n and the arrival rate λ_n in the case of $s=3$

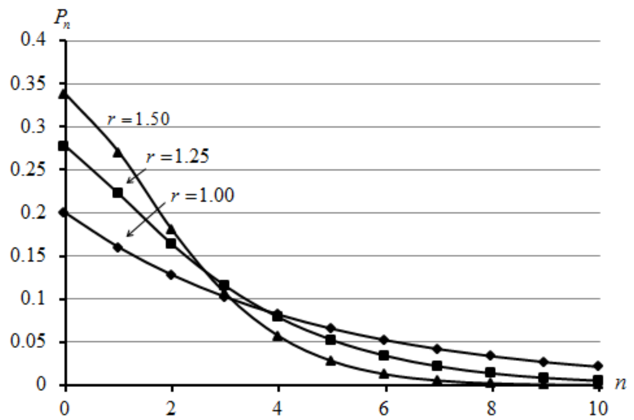

Figure 4: the relationship between the queue length n and the equilibrium state probability P_n in the case of $s = 1$

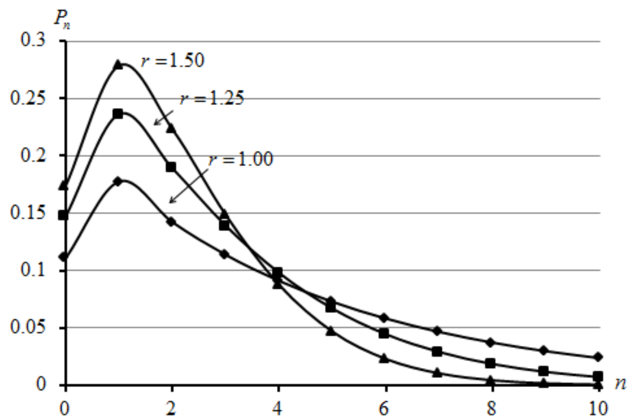

Figure 5: the relationship between the queue length n and the equilibrium state probability P_n in the case of $s = 2$

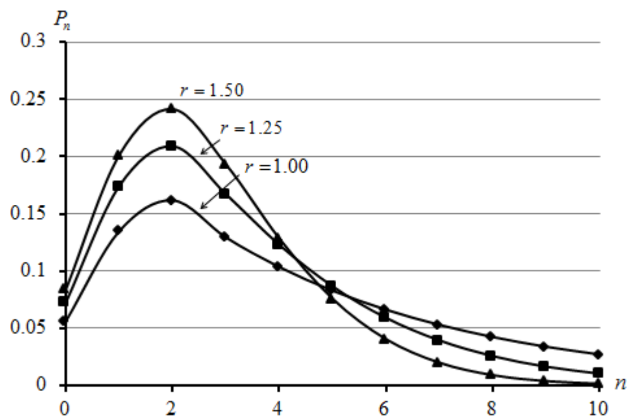

Figure 6: the relationship between the queue length n and the equilibrium state probability P_n in the case of $s = 3$

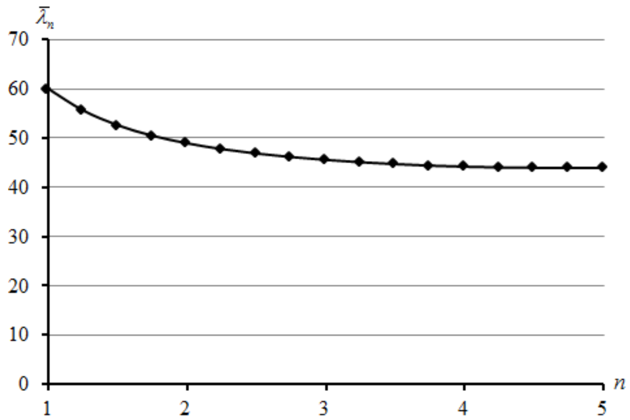

Figure 7: the relationship between the system parameter r and the expected arrival rate $\bar{\lambda}_n$ in the case of $s = 3$

arrival rate for each state of waiting consumers is also obtained under the consideration of balking. Then, we can evaluate the expected arrival rate in the M/M/s queueing system with balking.

Moreover, it has shown that the equilibrium state probability distribution based on the proposed statistical mechanics model involves both the equilibrium state probability distribution for the M/M/s queueing system without balking based on the traditional statistical equilibrium analysis and the equilibrium state probability distribution for the M/M/1 queueing system with balking, as respective special cases. Through the mathematical verification, it has been shown that the novel statistical mechanics model is applicable to various types of Markovian queueing systems. Therefore, we conclude that the novel statistical mechanics model in this study have general versatility to analyze Markovian queueing systems.

We would like to address the optimization problem of the M/M/s queueing system under the consideration of the balking phenomenon as a future study. In such a case, the estimation of the system parameter r based on the observation of the queueing system should be required. These issues will be with future challenges.

REFERENCES

Abou-El-Ata, M. O., and Hariri, A. M. A. (1992) The M/M/c/N Qqueue with Balking and Reneging. *Computers & operations research*, **19** (8), 713-716.

Arizono, I., Cui, Y., and Ohta, H. (1991) An Analysis of M/M/s Queueing Systems Based on the Maximum Entropy Principle, *Journal of the Operational Research Society*, **46** (2), 245-253.

Borzadaran, G. R. M. (2009) A Note on Maximum Entropy in Queueing Problems, *Economic Quality Control*, **24** (2), 263-267.

Chandler, D. (1987) *Introduction to Modern Statistical Mechanics*, Oxford University Press.

El-Sherbiny, A. A. (2012) The Truncated Heterogeneous Two-server Queue: M/M/2/N with Reneging and General Balk Function. *International Journal of Mathematical Archive*, **3** (7), 2745-2754.

Guiasu, S. (1986) Maximum Entropy Condition in Queueing Theory, *Journal of the Operational Research Society*, **37** (3), 293-301.

Jain, M. and Dhakad, M. R. (2003) Maximum Entropy Analysis for G/G/1 Queuing System, *IJE transactions A: Basics*, **16** (2), 163-170.

Jain, N. K., Kumar, R., and Som B. K. (2014) An M/M/1/N Queuing System with Reverse Balking, *American Journal of Operational Research*, **4** (2), 17-20.

János, S. (2011) *Basic Queueing Theory*, University of Debrecen: Faculty of Informatics.

Montazer-Haghighi, A., Medhi, J., & Mohanty, S. G. (1986) On a Multiserver Markovian Queueing System with Balking and Reneging. *Computers & operations research*, **13**(4), 421-425.

Prabhakar, B. (2003) Entropy and the Timing Capacity of Discrete Queues, *IEEE Transactions on Information Theory*, **49** (2), 357-370.

Singh, S. N. and Tiwari, S. B. (2013) An Application of Generalized Entropy in Queueing Theory, *Journal.of Applied Science and Engineering*, **16** (1), 99-103.

Tamura, K., Arizono, I., Kato, W., and Takemoto, Y. (2015) Proposal of Statistical Mechanics Mode for M/M/1 Queueing System with Balking, *Proc. of the 16th Asia Pacific Industrial Engineering and Management Systems Conference* (APIEMS 2015), CDROM.

Tribus, M., and McIrvine E. C. (1971) *Energy and information*, Scientific American.

Wang, K.-H., Chuang S.-L., and Pearn, W.-L. (2002) Maximum Entropy Analysis to the *N* Policy M/G/1 Queueing System with a Removable Server, *Applied Mathematical Modeling*, **26** (12), 1151-1162.