# Stochastic Formulations of Importance Measures and Their Extensions to Multi-State Systems

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**Abstract.** In the case of binary state reliability system, importance measures of components consisting a system, such as structural importance measure, Birnbaum importance measure, criticality importance measures and some other importance measures, have been proposed and used effectively for practical risk and safety problems. When we want to judge which component or factor should be maintained first to improve the system's performance, the importance measure suggests us that the most important component in a sense of an importance measure should be the first candidate. But these measures are not necessarily defined clearly from a stochastic theoretical point of view, and also extensions of them to the case of multi-state systems are not sufficiently achieved. In this paper, we show stochastically clear definitions of these importance measures with basic mathematical ideas behind them. The definitions do not need a usually assumed stochastic independence among components and then we may naturally extend the measures for the binary case to the multi-state case in various ways. An algorithm to give an extended Birnbaum importance measure from minimal state vectors which uniquely determine the structure function of the system is also given. And a calculation method of importance measures via a modular decomposition is also given.

**Keywords:** importance measure, modular decomposition, multi-state system

## 1. INTRODUCTION

Some notions of importance measures of components have been proposed and played a crucial role in reliability decision making. The well-known notions are structure (Barlow et al., 1975) and Birnbaum (Birnbaum, 1968, 1969), criticality (Bisanovic, et.al., 2013; Espiritu, et al., 2007), Barlow and Proschan's (Barlow, et al., 1974) importance measures, which are based on the concept of critical state vectors. Fussell-Vesely's importance measure (Fussell, 1975) and risk achievement and reduction worth (Cheok, et al., 1998) also play important roles in reliability decision making. These importance measures have been summarized by Kuo, et al.(2012), where we can find various definitions of importance measures based on the fruits by structural studies of a binary state system. Many authors have studied about binary state systems and these studies have been summarized by Barlow, et al.(1975). See also Mine(1959), Birnbaum, et al.(1965), Birnbaum, et al.(1961) and Esary, et al.(1963).

Systems and their components, however, could practically take many intermediate performance levels between perfectly functioning and complete failure states, and furthermore several states sometime can not be compared with each other. Mathematical studies about multi-state systems with totally ordered state spaces have been performed by many authors. See Barlow, et al.(1978), Griffith(1980), El-Neweihi, et al.(1978), Natvig(1982), Ohi, et al.(1983)(1984)(1984), Ohi(2010). Huang, et al.(2003) have extended a binary state consecutive *k*-out-of-*n* system. Natvig(2011), Lisnianski, et al.(2003), Lisnianski, et al.(2010) have summarized the work performed so far, and we may find examples of practical applications of multi-state systems. Levitin(2008)(2005)(2004) have extensively applied the universal generating function（UGF）method for solving reliability problems of multi-state systems and showed its effectiveness. UGF method was first proposed by Ushakov(1987)(2000) as a stochastic evaluation method of multi-state systems, and is especially thought to be effective for the stochastic analysis of a system hierarchically composed of modules like series-parallel or parallel-series systems. Ohi(2014) has generally given stochastic upper and lower bounds for system's stochastic performances via modular decompositions, which are convenient for systems designers and analysts. Furthermore, a model of partially ordered state spaces is required for the reliability analysis in a situation that we can not say for two states which state is good or not. Such a model has been recently examined and some useful stochastic evaluation methods have been proposed. See Levitin(2013), Yu, et al.(1994), and Ohi(2012)(2013)(2014)(2015).

We have some works about multi-state importance measures by Levitin, et al.(1999), Levitin, et al.(2003), Natvig(2011). Importance measures in the case of binary state systems, however, have not been sufficiently extended to multi-state systems.

In this paper, we first present stochastic formulations of Birnbaum and criticality importance measures in the binary case. We especially show the consistency of the magnitude relations of Birnbaum and criticality importance measures for a series-parallel system which is well observed system's structure in a practical situation. This consistency tells us that it does not matter whichever Birnbaum or criticality importance measure we use for judging the importance of components. These two importance measures are commonly defined on the basis of critical state vectors, which may be derived from minimal path and cut vectors. We show a basic algorithm for the derivation and a chain rule of multiplication via a modular decomposition.

Learning the binary case, we extend the Birnbaum importance measure in two ways to a multi-state system with totally ordered state spaces. The cardinal numbers of the state spaces are not necessarily the same. Our generalization is stochastically achieved and is based on the generalized critical state vectors, which are derived from the minimal and maximal state vectors corresponding to the minimal path and cut vectors of a binary state system, respectively. We also present that the chain rule of multiplication via a modular decomposition holds for a kind of Birnbaum importance measure. The chain rule is applied to a series-parallel multi-state system to give Birnbaum importance measures. Because of the limitation of the number of pages, we omit extensions of other importance measures as criticality, Fussell-Vesely's importance measures.

## 2. NOTATIONS

Finite sets $C=\{1,2,\cdots,n\}$, $\Omega_i$ $(i\in C)$ and $S$ are respectively the set of the components, the state space of the $i$-th component and the state space of the system. $\varphi$ is a mapping from $\Omega_C=\prod_{i\in C}\Omega_i$ to $S$. The precise definition of a multi-state system is presented in Definition3.

1. $\{x;A\}$ is the set of the element $x$ which satisfies the condition $A$.
2. For two sets $A$ and $B$,
$$A\setminus B=\{\ x\ \mid\ x\in A\ ,x\notin B\ \}.$$
3. $P(\ D\ \mid\ E\ )$ is the conditional probability of the event $D$ with respect to the event $E$.
4. For $A\subseteq C$, $\Omega_A=\prod_{i\in A}\Omega_i$.
5. An element $\boldsymbol{x}$ of $\Omega_C$ is precisely written as $x=(x_1,\cdots,x_n)$, where $x_i\in\Omega_i$ $(i=1,\cdots n)$.
6. Letting $\{\ B_j\ \mid\ 1\leqq j\leqq m\ \}$ be a partition of $A\subseteq C$, for $x_j\in\prod_{i\in B_j}\Omega_i$ $(1\leqq j\leqq m)$, $\boldsymbol{x}=(x_1,\cdots,x_m)$ is an element of $\Omega_A$ such that $P_{B_j}(\boldsymbol{x})=x_j$. Then for every $\boldsymbol{x}\in\Omega_A$ $(A\subseteq C)$,
$$\boldsymbol{x}=(\boldsymbol{x}^{B_1},\cdots,\boldsymbol{x}^{B_m}),$$
where $\boldsymbol{x}^{B_j}=P_{B_j}(\boldsymbol{x})$ $(j=1,\cdots,m)$. $P_{B_j}$ is the projection mapping from $\Omega_A$ to $\Omega_{B_j}$.
7. For $\boldsymbol{x}\in\Omega_A$ $(A\subseteq C)$, $(k_i,\boldsymbol{x})$ $(i\in A)$ denotes the $i$-th coordinate of $\boldsymbol{x}$ is $k$. The symbol $(\cdot_i,\boldsymbol{x})$ also denotes a state vector of $\Omega_{A\setminus\{i\}}$ given by deleting $x_i$ from $\boldsymbol{x}\in\Omega_A$ or a state vector $\boldsymbol{x}\in\Omega_{A\setminus\{i\}}$ attached with the empty $i$-th coordinate. When $A=C$, for $i\in C$,
$$(\cdot_i,\boldsymbol{x})=(x_1,\cdots,x_{i-1},\cdot\ ,x_{i+1},\cdots,x_n)\in\Omega_{C\setminus\{i\}}.$$
8. Every order is commonly denoted by the symbol $\leqq$. $a\leqq b$ means $b$ is greater than or equal to $a$.
9. For a subset $A$ of $S$, $\varphi^{-1}(A)$ is the inverse image of $A$ with respect to $\varphi$. The bracket is sometimes dropped out without any confusion.
10. For an element $x$ of an ordered set $W$, intervals $[x,\rightarrow)$ and $(\leftarrow,x]$ are defined as
$$[x,\rightarrow)\overset{def}{=}\{\,y\in W\ \mid\ y\geqq x\},\quad(\leftarrow,x]\overset{def}{=}\{\,y\in W\ \mid\ y\leqq x\}.$$
11. $MI(W)$ and $MA(W)$ denote the set of all the minimal and maximal elements of a finite ordered set $W$, respectively.
12. $\otimes$ means the product probability.

## 3. PRELIMINARY DEFINITIONS AND THEOREMS

**Definition 1.** (Definition of a multi-state system) A multi-state system is a triplet $(\prod_{i\in C}\Omega_i\,,S,\varphi)$ satisfying the following conditions.
(i) $C=\{1,2,\cdots,n\}$ denotes a set of the components consisting the system.
(ii) $\Omega_i$ $(i\in C)$ and $S$ are finite totally ordered sets, which denote the state spaces of the component $i$ and the system, respectively.
(iii) $\varphi$ is a surjection from $\prod_{i\in C}\Omega_i$ to $S$, called a structure function.
When $|\Omega_i|=2$ $(i\in C)$ and $|S|=2$ hold, the system is called a binary state system.

**Definition 2.** (Increasing system) A system $\varphi$ is called an increasing system, when the following condition is satisfied.
$$\forall\boldsymbol{x},\forall\boldsymbol{y}\in\Omega_C\ \text{such that}\ \boldsymbol{x}\leqq\boldsymbol{y},\varphi(\boldsymbol{x})\leqq\varphi(\boldsymbol{y}).$$

**Theorem 1.** *For an increasing system* $(\Omega_C,S,\varphi)$, *we have*
$$\forall\boldsymbol{x}\in\Omega_C,\varphi(\boldsymbol{x})=\max\ \{\ t\ \mid\ \exists\boldsymbol{a}\in MI(\varphi^{-1}[t,\rightarrow)),\boldsymbol{x}\geqq\boldsymbol{a}\ \}.\quad(1)$$

Theorem 1 shows us that the structure function $\varphi$ is uniquely determined by the family $\{MI(\varphi^{-1}[s,\rightarrow))\}_{s\in S}$ or $\{MA(\varphi^{-1}(\leftarrow,s])\}_{s\in S}$. In the binary case, $MI(\varphi^{-1}[1,\rightarrow))=MI(\varphi^{-1}(1))$ and $MA(\varphi^{-1}(\leftarrow,0])=MA(\varphi^{-1}(0))$ are respectively the sets of the minimal path and cut vectors.

**Definition 3.** (Modular decomposition) A partition $\{A_1,\cdots,A_m\}$ of $C$ is called a modular decomposition of an increasing system $(\Omega_C,S,\varphi)$, when there exist increasing systems $\left(\Omega_{A_j},S_j,\chi_j\right)$ $(j=1,\cdots,m)$ and $\left(\prod_{j=1}^m S_j\,,S,\psi\right)$ such that
$$\forall\boldsymbol{x}\in\Omega_C,\varphi(\boldsymbol{x})=\psi\big(\chi_1(\boldsymbol{x}^{A_1}),\cdots,\chi_m(\boldsymbol{x}^{A_m})\big).$$

## 4. BINARY STATE CASE

In this section we present the stochastic formulation of Birnbaum and criticality importance measures of a binary state system $(\Omega_C, S, \varphi)$. $P$ is a probability on $\Omega_C$.

### 4.1. Birnbaum and Criticality Importance Measures

We set

$$C_\varphi(i) \overset{def}{=} \{ \ \boldsymbol{x} \in \prod_{j \in C\{i\}} \Omega_j \ \mid \ \varphi(1_i, \boldsymbol{x}) = 1, \varphi(0_i, \boldsymbol{x}) = 0 \ \} \qquad (2)$$

A state vector of $C_\varphi(i)$ is called a critical state vector of the component $i$ and means a circumstance where the state of the component $i$ is critical to the system.

**Definition 4.** (Birnbaum importance measure) The Birnbaum importance measure of the component $i \in C$ is defined to be the probability $P^{C\setminus\{i\}}(C_\varphi(i))$.

Critical state vectors are determined by the minimal path vectors in $MI(\varphi^{-1}(1))$ and the minimal cut vectors in $MA(\varphi^{-1}(0))$.

**Theorem 2.** (*Determination of* $C_\varphi(i)$) *For a state vector* $\boldsymbol{x} \in \Omega_C$, $(\cdot_i, \boldsymbol{x})$ *is a critical state vector of the component* $i$ *if and only if*

$$\exists \boldsymbol{p} \in MI(\varphi^{-1}(1)), \exists \boldsymbol{k} \in MA(\varphi^{-1}(0)), \quad (\cdot_i, \boldsymbol{k}) \geqq (\cdot_i, \boldsymbol{x}) \geqq (\cdot_i, \boldsymbol{p}). \qquad (3)$$

*For these* $\boldsymbol{p}$ *and* $\boldsymbol{k}$, $p_i = 1$ *and* $k_i = 0$ *hold by the increasing property of* $\varphi$.

Theorem 2, which is equivalent to Theorem 1 of Meng(1966), gives us a convenient form for us to give an algorithm of determining our extended critical state vectors in the multi-state case. By this theorem, we have a determination procedure of $C_\varphi(i)$.

**STEP 1.** $MI \times MA_\varphi(i) \overset{def}{=} \{ \ (\boldsymbol{p}^{C\setminus\{i\}}, \boldsymbol{k}^{C\setminus\{i\}}) \ \mid \ (\cdot_i, \boldsymbol{k}) \geqq (\cdot_i, \boldsymbol{p}), \boldsymbol{p} \in MI(\varphi^{-1}(1)), \boldsymbol{k} \in MA(\varphi^{-1}(0)) \ \}$.

**STEP 2.** For $(\boldsymbol{p}, \boldsymbol{k}) \in MI \times MA_\varphi(i)$,

$$X_\varphi(i, \boldsymbol{p}, \boldsymbol{k}) \overset{def}{=} \{ \ (\cdot_i, \boldsymbol{x}) \ \mid \ (\cdot_i, \boldsymbol{k}) \geqq (\cdot_i, \boldsymbol{x}) \geqq (\cdot_i, \boldsymbol{p}) \ \}.$$

**STEP 3.** $C_\varphi(i) = \bigcup_{(\boldsymbol{p}, \boldsymbol{k}) \in MI \times MA_\varphi(i)} X_\varphi(i, \boldsymbol{p}, \boldsymbol{k})$.

By the total probability law, the Birnbaum importance measure of the component $i$ is decomposed as follows.

$$P^{C\setminus\{i\}}(C_\varphi(i)) = P(\{1_i\} \times C_\varphi(i) \ \mid \ \varphi = 1 \ ) \cdot P(\varphi = 1) + P(\{0_i\} \times C_\varphi(i) \ \mid \ \varphi = 0) \cdot P(\varphi = 0), \qquad (4)$$

where $\{1_i\} \times C_\varphi(i) = \{ \ (1_i, x) : \ x \in C_\varphi(i) \ \}$. Taking separately each term in the right hand side of (4), we have the following definition of two kinds of criticality importance measures.

**Definition 5.** (Criticality importance measure) The criticality importance measures of the component $i$ are defined in the two ways.

$$P(\{1_i\} \times C_\varphi(i) \ \mid \ \varphi = 1), \qquad (5)$$

$$P(\{0_i\} \times C_\varphi(i) \ \mid \ \varphi = 0). \qquad (6)$$

When the components are stochastically independent,

$$P(\{1_i\} \times C_\varphi(i) \ \mid \ \varphi = 1) = P^{C\setminus\{i\}}(C_\varphi(i)) \cdot \frac{P^i(1)}{P(\varphi = 1)}, \qquad (7)$$

$$P(\{0_i\} \times C_\varphi(i) \ \mid \ \varphi = 0) = P^{C\setminus\{i\}}(C_\varphi(i)) \cdot \frac{P^i(0)}{P(\varphi = 0)}. \qquad (8)$$

### 4.2. Modular Decomposition and Importance Measures

For a binary state system $(\Omega_C, S, \varphi)$, a modular decomposition $\{A_1, \cdots, A_m\}$ is supposed to be given. Furthermore, a probability $P$ on $\Omega_C$ is assumed to be the product probability of $P^{A_j}, j = 1, \cdots, m$. $\chi_j \circ P^{A_j}$ is the image probability of $P^{A_j}$ on $S_j$ by $\chi_j$ $(j = 1, \cdots, m)$, $\bigotimes_{j=1}^m \chi_j \circ P^{A_j}$ is the product probability on $\prod_{j=1}^m S_j$, and $\psi \circ (\bigotimes_{j=1}^m \chi_j \circ P^{A_j})$ is the image probability of $\bigotimes_{j=1}^m \chi_j \circ P^{A_j}$ on S by $\psi$. $\varphi \circ P = \psi \circ (\bigotimes_{j=1}^m \chi_j \circ P^{A_j})$ clearly holds. Importance measures of the modules $A_j, j = 1, \cdots, m$ in the system $\psi$ is defined by the probability $\bigotimes_{j=1}^m \chi_j \circ P^{A_j}$. We use a symbol $j_i$ denoting the index number of the module which contains the component $i$.

We suppose module 1 contains the component 1 without loss of generality.

**Theorem 3.** *For the Birnbaum and criticality importance measures, we have the following chain rule via a modular decomposition.*

$$P^{C\setminus\{1\}}(C_\varphi(1)) = P^{A_1\setminus\{1\}}(C_{\chi_1}(1)) \times \bigotimes_{j=2}^m \chi_j \circ P^{A_j}(C_\psi(1)), \qquad (9)$$

$$P(\{1_1\} \times C_\varphi(1) \ \mid \ \varphi = 1) = \frac{P^{A_1}(\{1_1\} \times C_{\chi_1(1)})}{P^{A_1}(\chi_1 = 1)} \cdot \frac{\bigotimes_{j=1}^m \chi_j \circ P^{A_j}(\{1_1\} \times C_\psi(1))}{\bigotimes_{j=1}^m \chi_j \circ P^{A_j}(\psi = 1)}, \qquad (10)$$

$$P(\{0_1\} \times C_\varphi(1) \ \mid \ \varphi = 0) = \frac{P^{A_1}(\{0_1\} \times C_{\chi_1}(1))}{P^{A_1}(\chi_1 = 0)}$$

$$\cdot \frac{\bigotimes_{j=1}^{m} \chi_j \circ P^{A_j}(\{0_1\} \times C_\psi(1))}{\bigotimes_{j=1}^{m} \chi_j \circ P^{A_j}(\psi = 0)}. \tag{11}$$

The equality (9) tells us that the next equality holds.

· **[** the Birnbaum importance measure of the component 1 in the system $\varphi$ **]**
= **[** the Birnbaum importance measure of the component 1 in the system $\chi_1$ **]**
× **[** the Birnbaum importance measure of the module 1 in the system $\psi$ **]**

From (10) and (11) we have the similar equalities about the criticality importance measures.

· **[** the criticality importance measure of the component 1 in the system $\varphi$ **]**
= **[** the criticality importance measure of the component 1 in the system $\chi_1$ **]**
× **[** the criticality importance measure of the module 1 in the system $\psi$ **]**

### 4.3. Importance Measures of a Series-Parallel System

In this section, we show a consistency of the magnitude relations among importance measures of the components of a series-system which is well observed in practical situations. See Figure 1. The system may be considered to have a modular decomposition $\{A_i, i = 1, \cdots, m\}$, where

$$A_i = \{(i,1), \cdots, (i,n_i)\}.$$

Each module is a parallel system and the organising structure $\psi$ is a series system. The components are assumed to be stochastically independent and doubly indexed.

The Birnbaum and criticality importance measures of the component $(1,1)$ are respectively given by the chain rule as follows.

$$P^{C\setminus\{(1,1)\}}\big(C_\varphi((1,1))\big) = P^{A_1\setminus\{(1,1)\}}\{(\cdot_{(1,1)}, \mathbf{0})\} \times \bigotimes_{j=2}^{m} \chi_j \circ P^{A_j}\{(\cdot_1, \mathbf{1})\}.$$

$$P\big(\{1_{(1,1)}\} \times C_\varphi((1,1)) \mid \varphi = 1\big) = \frac{P^{A_1}\{(1_{(1,1)}, \mathbf{0})\}}{P^{A_1}(\chi_1 = 1)},$$

$$P\big(\{0_{(1,1)}\} \times C_\varphi((1,1)) \mid \varphi = 0\big) = \frac{\bigotimes_{j=1}^{m} \chi_j \circ P^{A_j}\{(0_1, \mathbf{1})\}}{\bigotimes_{j=1}^{m} \chi_j \circ P^{A_j}(\psi = 0)}.$$

By the above equalities, the following two inequalities (i) and (ii) are equivalent with each other and also (iii) and (iv) are equivalent, which mean that for every series-parallel system, it does not matter which importance measure you use. This assertion is still true for a system which is hierarchically composed of series and parallel modules.

(i) Birnbaum importance measure of the component (1,1)
≦ Birnbaum importance measure of the component (2,1)

(ii) criticality importance measure of the component (1,1)
≦ criticality importance measure of the component (2,1)

(iii) Birnbaum importance measure of the component (1,1)
≦ Birnbaum importance measure of the component (1,2)

(iv) criticality importance measure of the component (1,1)
≦ criticality importance measure of the component (1,2)

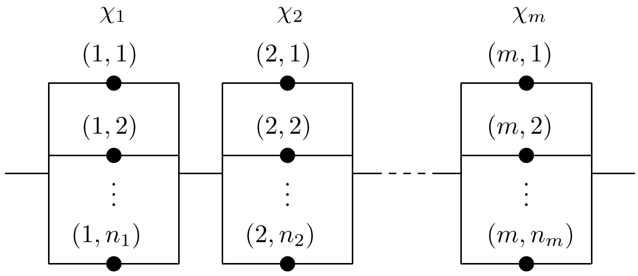


Figure 1: A series-parallel system composed of components doubly indexed as (*i*,*j*), where *i* is the number of the module and *j* is the number of the component in the module *i*.

## 5. MULTI-STATE CASE

For a multi-state system $(\Omega_C, S, \varphi)$, we define some kinds of critical state vectors of a component $i \in C$ which correspond to (2) for a binary state system. For $k \in \Omega_i$ and $s \in S$,

$$C_\varphi(i,k;s) \overset{def}{=} \{ \ (\cdot_i, \boldsymbol{x}) \in \Omega_{C\setminus\{i\}} \mid \varphi(k_i, \boldsymbol{x}) \geqq s \text{ and } \varphi((k-1)_i, \boldsymbol{x}) \leqq s-1 \ \}. \tag{12}$$

For states $k, l \in \Omega_i$, and $s, t \in S$,

$$C_\varphi(i,k,l;s,t) \overset{def}{=} \{ \ (\cdot_i, \boldsymbol{x}) \in \Omega_{C\setminus\{i\}} \mid \varphi(k_i, \boldsymbol{x}) = s, \ \varphi(l_i, \boldsymbol{x}) = t \ \}. \tag{13}$$

Clearly the following equality holds.

$$C_\varphi(i,k;s) = \bigcup_{u \leqq s-1, s \leqq v} C_\varphi\,(i, k-1, k \ ; u, v). \tag{14}$$

**Theorem 4.** (*Determination of* $C_\varphi(i,k;s)$) *For the state* $k$ *of the component* $i$ *and the state* $s \in S$, $(\cdot_i, \boldsymbol{x}) \in C_\varphi(i,k;s)$ *holds if and only if*

$$\exists (k_i, \boldsymbol{a}) \in MI\big(\varphi^{-1}[s, \rightarrow)\big),$$
$$\exists ((k-1)_i, \boldsymbol{b}) \in MA(\varphi^{-1}(\leftarrow, s-1]),$$
$$(k_i, x) \geqq \boldsymbol{a}, \qquad ((k-1)_i, x) \leqq \boldsymbol{b}.$$

Theorem 4 shows us how the critical state vectors of $C_\varphi(i,k;s)$ may be determined by the minimal and maximal state vectors by which the system is uniquely determined.

<u>Basic Algorithm for Determining</u> $C_\varphi(i,k;s)$

**STEP 1.** $MI \times MA_\varphi(i,k;s) \overset{def}{=} \{ \ (\boldsymbol{a}, \boldsymbol{b}); \ \boldsymbol{a}^{C\setminus\{i\}} \leqq \boldsymbol{b}^{C\setminus\{i\}},$

$a_i = k, b_i = k-1, \boldsymbol{a} \in MI(\varphi^{-1}[s, \rightarrow)),$
$\boldsymbol{b} \in MA(\varphi^{-1}(\leftarrow, s-1]) \}$,
**STEP 2**. For $(\boldsymbol{a}, \boldsymbol{b}) \in MI \times MA_\varphi(i,k;s)$,
$X_\varphi(i,k;s;\boldsymbol{a},\boldsymbol{b}) \overset{def}{=} \{ \ \boldsymbol{x} \in \Omega_{C\backslash\{i\}} \mid \boldsymbol{a}^{C\backslash\{i\}} \leqq \boldsymbol{x} \leqq \boldsymbol{b}^{C\backslash\{i\}} \ \}$,
**STEP 3**. $C_\varphi(i,k;s) = \bigcup_{(\boldsymbol{a},\boldsymbol{b}) \in MI \times MA_\varphi(i,k;s)} X_\varphi(i,k;s;\boldsymbol{a},\boldsymbol{b})$.

**Theorem 5.** (*Determination of* $C_\varphi(i,k,l;s,t)$) *Assuming* $k < l$ *and* $s < t$ *without loss of generality,* $(\cdot_i, \boldsymbol{x}) \in C_\varphi(i,k,l;s,t)$ *holds if and only if the following conditions hold.*

$$\exists \boldsymbol{a} \in MI(\varphi^{-1}(s)), \exists \boldsymbol{b} \in MA(\varphi^{-1}(s)),$$
$$\exists \boldsymbol{c} \in MI(\varphi^{-1}(t)), \exists \boldsymbol{d} \in MA(\varphi^{-1}(t)),$$
$$\boldsymbol{a} \leqq \boldsymbol{b}, \boldsymbol{c} \leqq \boldsymbol{d}, a_i \leqq k \leqq b_i < c_i \leqq l \leqq d_i,$$
$$(\cdot_i, \boldsymbol{c}) \leqq (\cdot_i, \boldsymbol{b}), (\cdot_i, \boldsymbol{a}) \leqq (\cdot_i, \boldsymbol{d}),$$
$$\boldsymbol{a} \leqq (k_i, \boldsymbol{x}) \leqq \boldsymbol{b}, \boldsymbol{c} \leqq (l_i, \boldsymbol{x}) \leqq \boldsymbol{d}.$$

Basic Algorithm for Determining $C_\varphi(i,k,l;s,t)$
**STEP 1**. $MI \times MA_\varphi(i,k,l;s,t) \overset{def}{=} \{ \ (\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{d}) \mid$
$\boldsymbol{a} \in MI(\varphi^{-1}(s)), \boldsymbol{b} \in MA(\varphi^{-1}(s)),$
$\boldsymbol{c} \in MI(\varphi^{-1}(t)), \boldsymbol{d} \in MA(\varphi^{-1}(t)),$
$\boldsymbol{a} \leqq \boldsymbol{b}, \boldsymbol{c} \leqq \boldsymbol{d}, a_i \leqq k \leqq b_i < c_i \leqq l \leqq d_i,$
$(\cdot_i, \boldsymbol{c}) \leqq (\cdot_i, \boldsymbol{b}), (\cdot_i, \boldsymbol{a}) \leqq (\cdot_i, \boldsymbol{d}) \ \}$.
**STEP 2**. For $(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{d}) \in MI \times MA_\varphi(i,k,l;s,t)$
$X_\varphi(\boldsymbol{i},\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{d}) \overset{def}{=} \{ \ x \in \Omega_{C\backslash\{i\}} \mid (\cdot_i, \boldsymbol{a}) \leqq (\cdot_i, \boldsymbol{x}) \leqq (\cdot_i, \boldsymbol{b}),$
$(\cdot_i, \boldsymbol{c}) \leqq (\cdot_i, \boldsymbol{x}) \leqq (\cdot_i, \boldsymbol{d}) \ \}$,
**STEP 3**. $C_\varphi(i,k,l;s,t)$
$$= \bigcup_{(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{d}) \in MI \times MA_\varphi(i,k,l;s,t)} X_\varphi(i,k,l;s,t).$$

**Example 1.** (Critical state vectors of a restricted series system) Assuming the state spaces of the components and the system to be

$$\Omega_i = S = \{0,1,2,\cdots,N\}, i = 1,\cdots,n,$$

we examine the critical state vectors of a series system of which structure function is give as

$$\varphi(x_1,\cdots,x_n) = \min_{1 \leqq i \leqq n} x_i.$$

The minimal and maximal state vectors are given as

$$s \in S, \quad MI_\varphi(s) = \{(s,\cdots,s)\},$$
$$MA_\varphi(s) = \{(\overset{i}{\check{s}}, \boldsymbol{N}) \mid i = 1,2,\cdots,n \ \},$$

where $(\overset{i}{\check{s}}, \boldsymbol{N}) = (N,\cdots,N,\overset{i}{\check{s}},N,\cdots,N)$, a state vector of which *i*-th coordinate is $s$ and others are *N*. Following the basic algorithm, we have

$$C_\varphi(i,s;s) = \{\boldsymbol{x} \in \Omega_{C\backslash\{i\}} \mid x_j \geqq s, \ j = 1,\cdots,n, j \neq i \ \} \tag{15}$$
$$C_\varphi(i,k;s) = \phi, \quad k \neq s.$$

The stricter critical state vectors of this series system may be given as follows. Assuming $s < t$,

$$C_\varphi(i,k,l;s,t) = \begin{cases} \{ \ \boldsymbol{x} \in \Omega_{C\backslash\{i\}} \mid x_j \geqq t, j = 1,\cdots,n, j \neq i \ \}, & k = s, l = t \\ \phi & \text{otherwise} \end{cases} \tag{16}$$

More general definition of a series and a parallel system may be found in Ohi (1983)(2013).

### 5.1. Extension of Birnbaum Importance Measure

Now assuming a probability $P$ to be endowed with $\Omega_C$, we extend the Birnbaum importance measure of Definition 2 to the multi-state case.

**Definition 6.** (Definition of Birnbaum importance measures)
(i) For states $k \in \Omega_i$ and $s \in S$, $P^{C\backslash\{i\}}(C_\varphi(i,k;s))$ is called the $(i,k;s)$-Birnbaum importance measure.
(ii) For two states $k$ and $l$ of $\Omega_i$ such that $k < l$ and a system's state $s$ and $t$ such that $s < t$, $P^{C\backslash\{i\}}(C_\varphi(i,k,l;s,t))$ is called the $(i,k,l;s,t)$-Birnbaum importance measure.

$(i,k;s)$-Birnbaum importance measure means the degree of critical contribution of the state $k$ of the component $i$ to the system's state $s$. $(i,k;s)$-Birnbaum importance measure is shown in Natvig(2011) for the totally ordered state spaces having the same cardinal number. In this paper, the sameness of the cardinal numbers is not assumed.

The chain rule of multiplication via a modular decomposition does not generally hold for $(i,k;s)$-Birnbaum importance measure, but holds for $(i,k,l;s,t)$-Birnbaum importance measure. Then, noticing

$$P^{C\backslash\{i\}}(C_\varphi;s)) = \sum_{u \leqq s-1, s \leqq v} P^{C\backslash\{i\}}(C_\varphi(i,k-1,k;u,v)) \tag{17}$$

by (14), this strict importance measure can be used complementary for obtaining the $(i,k;s)$-Birnbaum importance measure.

**Example 2.** (Continuation of Example 1) For the series system, the $(i,k;s)$-Birnbaum importance measure is given as follows.

$$P^{C\backslash\{i\}}(C_\varphi(i,s;s)) = P^{C\backslash\{i\}}\{\boldsymbol{x} \in \Omega_{C\backslash\{i\}} \mid x_j \geqq s, j = 1,\cdots,n, j \neq i \ \}$$
$$P^{C\backslash\{i\}}(C_\varphi(i,k;s)) = 0, \quad k \neq s.$$

s functioning.
$(i,k,l;s,t)$-Birnbaum importance measure is given as the following.

$$P^{C\backslash\{i\}}(C_\varphi(i,s,t;s,t)) = P^{C\backslash\{i\}}\{ \ \boldsymbol{x} \in \Omega_{C\backslash\{i\}} \mid x_j \geqq t, j = 1,\cdots,n, j \neq i \ \},$$
$$P^{C\backslash\{i\}}(C_\varphi(i,k,l;s,t)) = 0, \quad k \neq s, \text{or } \ l \neq t.$$

### 5.2. Chain Rule for Extended Birnbaum Importance Measure via Modular Decomposition

We suppose $\left(\prod_{j\in A_i}\Omega_j, S_i, \chi_i\right)$ $(i=1,\cdots,m)$ and $(\prod_{i=1}^{m} S_i, S, \psi)$ to be a modular decomposition of a system $(\prod_{i\in C}\Omega_i, S, \varphi)$ having a probability $P$ on $\Omega_C$. The probability $P$ is, furthermore, assumed to be the product probability of $P^{A_i}$ which is the restriction of $P$ to $\Omega_{A_i}$, $i=1,\cdots,n$.

We have the following relation among critical state vectors via the modular decomposition.

$$C_\varphi(i,k,l;s,t) = \bigcup_{s_{j_i},t_{j_i}\in S_{j_i}, s_{j_i}<t_{j_i}} C_{\chi_{j_i}}(i,k,l;s_{j_i},t_{j_i}) \times (\chi_1,\cdots,\chi_{j_i-1},\chi_{j_i+1},\cdots,\chi_n)^{-1}\left(C_\psi(j_i,s_{j_i},t_{j_i};s,t)\right) \quad (18)$$

where $j_i$ is the index number of the module to which the component $i$ belongs. Taking the probability of the both side of (18), we have the next theorem.

**Theorem 6.** *For the Birnbaum importance measures of systems of a modular decomposition, we have the following multiplicative relation.*

$$\begin{aligned}&P^{C\backslash\{i\}}\left(C_\varphi(i,k,l;s,t)\right)\\&= \sum_{s_{j_i},t_{j_i}\in S_{j_i}, s_{j_i}<t_{j_i}} P^{A_{j_i}\backslash\{i\}}\left(C_{\chi_{j_i}}(i,k,l;s_{j_i},t_{j_i})\right)\\&\quad\times(\chi_1,\cdots,\chi_{j_i-1},\chi_{j_i+1},\cdots,\chi_n)\\&\quad\circ P^{C\backslash A_{j_i}}\left(C_\psi(j_i,s_{j_i},t_{j_i};s,t)\right).\end{aligned} \quad (19)$$

The first and the second term in the summation of the right hand side of (19) are respectively the $(i,k,l;s_{j_i},t_{j_i})$-Birnbaum importance and $(j_i,s_{j_i},t_{j_i};s,t)$ -Birnbaum importance.

**Example 3.** (Series-parallel system) Using double index, we examine Birnbaum importance measures of a multi-state series-parallel system of which structure function is given as follows,

$$\varphi(\boldsymbol{x}^1,\cdots,\boldsymbol{x}^m) = \min_{1\leqq i\leqq m}\max_{1\leqq j\leqq n_i} x_{(i,j)}, \quad (20)$$

where we suppose $\Omega_{(i,j)} = S = \{0,1,\cdots,N\}$, $1\leqq i\leqq m, 1\leqq j\leqq n_i$ and $\boldsymbol{x}^i$ denotes

$$\boldsymbol{x}^i = (x_{(i.1)},\cdots,x_{(i,n_i)}), i=1,\cdots,m.$$

(20) means that the system is composed of $m$ modules each of which is a parallel system and the organising system is a series system, formally

$$A_i = \{\ (i,1),\cdots,(i,n_i)\ \}, i=1,\cdots,m,$$

$$\chi_i: \prod_{1\leqq j\leqq n_i}\Omega_j \to S_i = \{1,\cdots,N\}, \qquad \chi_i(\boldsymbol{x}^i) = \max_{1\leqq j\leqq n_i} x_{(i,j)},$$

$$\psi: \prod_{i=1}^{m} S_i \to S, \qquad \psi(s_1,\cdots,s_m) = \min_{1\leqq i\leqq m} s_i.$$

Then for the component $(1,1)$, without loss of generality, we have Birnbaum importance measures as follows.

$$\begin{aligned}&C_\varphi((1,1),s,t;s,t)\\&= C_{\chi_1}((1,1),s,t;s,t)\\&\quad\times(\chi_2,\cdots,\chi_m)^{-1}(C_\psi(1,s,t;s,t)),\\&C_\varphi((1,1),s;s)\\&= C_{\chi_1}((1,1),s;s)\\&\quad\times(\chi_2,\cdots,\chi_m)^{-1}(C_\psi(1,s;s)).\end{aligned}$$

$$\begin{aligned}&P^{C\backslash\{(1,1)\}}\left(C_\varphi((1,1),s,t;s,t)\right)\\&= P^{A_1\backslash\{(1,1)\}}\left(C_{\chi_1}((1,1),s,t;s,t)\right)\\&\quad\times(\chi_2,\cdots,\chi_m)\\&\quad\circ P^{C\backslash A_1}(C_\psi(1,s,t;s,t)),\\&P^{C\backslash\{(1,1)\}}\left(C_\varphi((1,1),s;s)\right) = P^{A_1\backslash\{(1,1)\}}\left(C_{\chi_1}((1,1),s;s)\right)\\&\quad\times(\chi_2,\cdots,\chi_m)\\&\quad\circ P^{C\backslash A_1}\left(C_\psi(1,s;s)\right),\end{aligned}$$

which denotes the Birnbaum importance measure of the system $\varphi$ is the multiplication of the Birnbaum importance measures of the module and the organizing system.

### 5.3. Overall Birnbaum Importance Measure

We notice that the $(i,k;s)$ -Birnbaum importance measure $P^{C\backslash\{i\}}\left(C_\varphi(i,k;s)\right)$ depends on the component $i$'s state $k$. Taking the summation of this probability with respect to $k\in\Omega_i$, we may define an overall importance measure of the component $i$ to some system's state greater than or equal to $s$. Noticing

$$s\neq t,\ \ C_\varphi(i,k;s)\cap C_\varphi(i,k;t) = \phi,$$

we have

$$P^{C\backslash\{i\}}\left(\cup_{k\in\Omega_i} C_\varphi(i,k;s)\right) = \sum_{k\in\Omega_i} P^{C\backslash\{i\}}\left(C_\varphi(i,k;s)\right).$$

We call this importance measure to be $(i;s)$ -importance measure.

For a binary state system, importance measure is defined for the good(operating) and bad(failure) states. For a multi-state system, when we consider an overall importance measure of a component, we should also define good and bad states. For the totally ordered state space, good states are those which are greater than or equal to some state $s$ and other states are bad. The threshold state $s$ is determined from a practical point of view.

We remain the precise examinations of this overall importance measure for future work.

## 6. CONCLUSION

In this paper, we have shown stochastic definitions of the Birnbaum and criticality importance measures of binary state systems. The definition of the former importance measure based on the critical state vector does not need an assumption of stochastic independence among components. Under the independence assumption, we may derive the Birnbaum and criticality importance measure by the differential calculation with respect to the reliability of a component.

$$P^{C\setminus\{i\}}\left(C_{\varphi}(i)\right)=\frac{\partial h(\boldsymbol{p})}{\partial p_i},$$
$$P\left(\{1_i\}\times C_{\varphi}(i)\middle|\varphi=1\right)=\frac{\partial \log h(\boldsymbol{p})}{\partial \log p_i},$$
$$P\left(\{0_i\}\times C_{\varphi}(i)\middle|\varphi=0\right)=\frac{\partial \log h(\boldsymbol{q})}{\partial \log q_i},$$
$$\boldsymbol{p}=(p_1,\dots,p_n),\boldsymbol{q}=(q_1,\dots,q_n),p_i+q_i=1,$$

where $p_i$ is the reliability of the component *i*. The stochastic definition presented in this paper is more general and easily extended to the multi-state case.

In this paper, for the binary state systems, we have examined the relations between Birnbaum and criticality importance measures and shown the consistency of the magnitude relations of the importance measures for a series-parallel system which is well observed in practical situations. This consistency means that Birnbaum and criticality importance measures have no difference with each other from the practical point of view. Furthermore, we have presented for the importance measures a chain rule of multiplication via a modular decomposition. The chain rule shows us that the importance measure of a system may be obtained by stacking up the importance measures of the module and the organizing system in a way of multiplication, which is thought to be convenient for designers.

In this paper, following the discussion of the binary state systems, we extended the Birnbaum importance measure to the multi-state system, of which state spaces are totally ordered sets, but the cardinal numbers of them are not necessarily the same. The extension is based on the two kinds of critical state vectors, following which we have given three kinds of Birnbaum importance measures, strict, soft and overall ones. The latter two measures are derived from the strict one. For the strict one, a chain rule via a modular decomposition holds.

We have not examined stochastically other importance measures as criticality importance measure, Fussell-Vesely's importance measure, risk achievement and reduction worth, Barlow-Proschan's importance measure in a multi-state context. Especially, the dynamics of components should be essentially incorporated into the definition of an importance measure. Some authors, Barlow, et al.(1974), Natvig(2011), have tried to propose notions of importance measures incorporated with time development of components. Examinations of these problems in general partially ordered state spaces are remained for future work.

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