Modified Replacement Overtime Policy for Shock and Damage Model

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Abstract This paper proposes an extended model of the replacement overtime policy for a cumulative damage model. We consider an operating unit which suffers some damage due to shocks. It is assumed that the total damage is additive, and the unit fails when the total damage has exceeded a prespecified level. We suppose that the unit is replaced at $\boldsymbol{N}$th ($\boldsymbol{N = 1, 2, \ldots}$) shock over the time $\boldsymbol{T}$ or at failure, whichever occurs first. That is, we start to observe occurrence of shocks after time $\boldsymbol{T}$. For such a model, we obtain the mean time to replacement and the expected costs rate, and discuss the optimal number of $\boldsymbol{N}$ which minimizes the expected cost rate when shocks occur in a Poisson process. Further, numerical examples are given, and suitable discussions are made.

Keywords: Overtime Policy, Cumulative Damage Process, Shock Model, Replacement Policy

1. INTRODUCTION

We propose an extended model of the replacement overtime policy for equipment management of shock and damage models. In recent years, equipment management has become more important to complete projects such as software development rapidly, safety and accurately. Furthermore, the equipment has become more complexity, and more difficult to check the state of the equipment by looking the appearance. We consider therefore a case of that the equipment is replaced at a completion of uses to avoid interruption of work on the way of using cycles. Such a model is called as *maintenance overtime policy* (Nakagawa and Zhao., 2015). Furthermore, we consider assumptions that the equipment has damage at every use, and fails when the total damage has exceeded a prespecified level. Such a model is called as *cumulative damage model* (Nakagawa, 2006).

We propose a maintenance policy which extends maintenance overtime policy for cumulative damage model. It is reasonable for such equipment to decide a maintenance such as scheduled time or number of shocks to maintain or replace the equipment. We treat a case that we cannot maintain the equipment until the scheduled time. One example is a rental of equipment with some reservations. For such a case, it is one way that the equipment is maintained or replaced at prespecified number of use over scheduled time.

There have many studies of maintenance policies using reliability theory (Barlow and Proschan, 1965; Nakagawa, 2015). The maintenance models that the unit is replaced at a random working time are studied (Nakagawa, 2014; Chen et.al., 2010). Maintenance overtime policies where the unit is replaced at a first time of completion of works over planned time have been discussed (Nakagawa and Zhao., 2015; Zhao et.al., 2013; Zhao et.al., 2014). The cumulative damage model have also many studies in reliability theory (Stallmeyer, 1968; Bogdanoff. et.al., 1985; Nakagawa, 2006).

In this paper, we consider an extended replacement overtime policy for a cumulative damage to maintain an operating unit. The unit which is used for random times and suffers some damage due to shocks. It is assumed that the unit fails when the total damage has exceeded a prespecified level. The total damage is additive, and the amount of damage cannot be investigated. We assume that the unit is replaced at Nth ($N = 1,2,\ldots$) shock over planned time T or at failure, whichever occurs first. Figure 1 shows the process of the model when the unit is replaced at Nth shock over time T. Figure 2 shows the process when the units fails and is replaced. That is, we start to observe occurrences of shocks after time T, and introduce a replacement cost and monitoring cost.

For such a model, we obtain the expected costs rate and discuss optimal policies which minimize it. Section 2 shows the assumptions and notations of the model, and obtains the mean time to replacement and the expected cost rate. Section 3 discusses optimal number N and time T which minimize the expected cost rate when shocks occur in a Poisson process. Sections 4 gives numerical examples of optimal N and T when each damage is exponential. We discuss the tendencies for several parameters in numerical examples.

2. ASSUMPTIONS

We make following assumptions of the replacement policy for the cumulative damage model:

(i) Let X_j be a random variable that denotes a sequence of interval times between successive shocks with an identical distribution $F(t) \equiv \Pr\{X_j \le t\}$ $(j = 1,2,\ldots)$ and finite mean $\mu \equiv \int_0^t \bar{F}(u)\,du$. A density function of $F(t)$ is $f(t) \equiv dF(t)/dt$, i.e., $F(t) = \int_0^t f(u)\,du$, and the failure rate is $h(t) \equiv f(t)/\bar{F}(t)$, where $\bar{\Phi}(t) \equiv 1 - \Phi(t)$ for any function $\Phi(t)$. The failure rate increases strictly with t from $h(0)$ to $h(\infty)$. The j-fold Stieltjes convolution of $F(t)$ is $F^{(j)}(t) \equiv \Pr\{X_1 + X_2 + \cdots + X_j \le t\}$ ($j = 1,2,\ldots$) and $F^{(0)}(t) \equiv 1$ for $t \ge 0$.

(ii) Let W_j be a random variable that denotes the damage produced by the jth shock, where $W_0 \equiv 0$, with a cumulative distribution $G(t) \equiv \Pr\{W_j \le t\}$ ($j = 1,2,\ldots$). The j-fold Stieltjes convolution of $G(t)$ is $G^{(j)}(t) \equiv \Pr\{W_1 + W_2 + \cdots + W_j \le t\}$ ($j = 1,2,\ldots$) and $G^{(0)}(t) \equiv 1$ for $t \ge 0$.

(iii) Let $N(t)$ denote the random variable which is the total number of shocks up to time t ($t \ge 0$). Then, define a random variable

$$Z(t) \equiv \sum_{j=0}^{N(t)} W_j\ ,$$

which represents the total damage at time t. It is assumed that the unit fails when the total damage has exceeded a prespecified level K ($0 < K < \infty$).

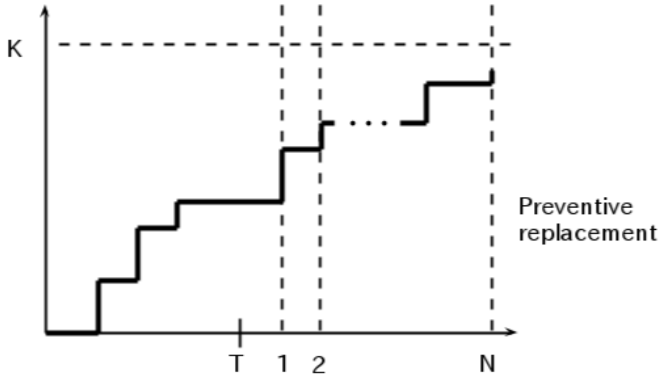

Figure 1: Process for preventive replacement.

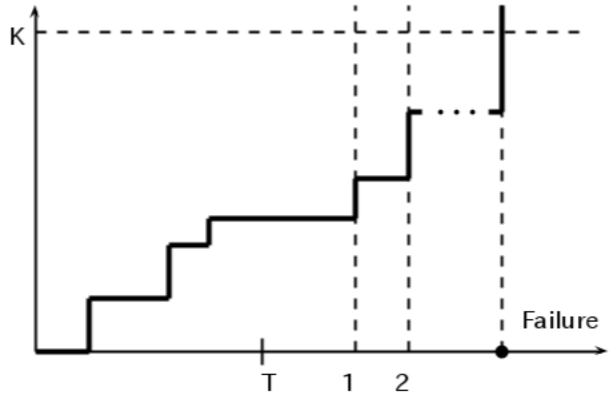

Figure 2: Process for failure.

(iv) The unit is replaced at Nth ($N = 1,2,\ldots$) shock over time T or at failure, whichever occurs first.

(v) Cost c_F is a replacement cost when the unit fails, and cost c_N ($c_F > c_N$) is a replacement cost when the unit is replaced at Nth shock over time T.

Form the above assumptions, we obtain the expected cost rate. The probability that the unit is replaced at shock N over time T is

$$\sum_{j=0}^{\infty} G^{(j+N)}(K) \int_0^T \{\int_{T-t}^{\infty} [\int_u^{\infty} dF^{(N-1)}(v-u)] \times dF(u)\} dF^{(j)}(t)$$

$$= \sum_{j=0}^{\infty} [F^{(j)}(T) - F^{(j+1)}(T)] G^{(j+N)}(K), \tag{1}$$

and the probability that it is replaced at failure is

$$\sum_{j=0}^{\infty} F^{(j+1)}(T)[G^{(j)}(K) - G^{(j+1)}(K)] + \sum_{j=0}^{\infty} [F^{(j)}(T) - F^{(j+1)}(T)][G^{(j)}(K) - G^{(j+N)}(K)], \tag{2}$$

where (1) + (2) = 1. The mean time to replacement is

$$\sum_{j=0}^{\infty} G^{(j+N)}(K) \int_0^T \{\int_{T-t}^{\infty} [\int_u^{\infty} (t+v) dF^{(N-1)}(v-u)] \times dF(u)\} dF^{(j)}(t)$$
$$+ \sum_{j=0}^{\infty} [G^{(j)}(K) - G^{(j+1)}(K)] \int_0^T t\, dF^{(j+1)}(t)$$
$$+ \sum_{j=0}^{\infty} \sum_{i=0}^{N-1} [G^{(j+i)}(K) - G^{(j+i+1)}(K)] \times \int_0^T \{\int_{T-t}^{\infty} [\int_u^{\infty} (t+v) dF^{(i)}(v-u)] dF(u)\} dF^{(j)}(t)$$
$$= \sum_{j=0}^{\infty} G^{(j+N)}(K) \int_0^T (\int_{T-t}^{\infty} \{\int_u^{\infty} [1 - F^{(N-1)}(v-u)] dv\} \times dF(u)) dF^{(j)}(t) + \mu \sum_{j=0}^{\infty} F^{(j)}(T) G^{(j)}(K)$$
$$+ \sum_{j=0}^{\infty} \sum_{i=0}^{N-1} [G^{(j+i)}(K) - G^{(j+i+1)}(K)] \times \int_0^T (\int_{T-t}^{\infty} \{\int_u^{\infty} [1 - F^{(i)}(v-u)] dv\} dF(u)) dF^{(j)}(t)$$
$$= \mu \sum_{j=0}^{\infty} [F^{(j)}(T) - F^{(j+1)}(T)] \sum_{i=1}^{N+j-1} G^{(i)}(K). \quad (3)$$

Therefore, the expected cost rate is

$$C(N,T) = \frac{c_F - (c_F - c_N) \sum_{j=0}^{\infty} [F^{(j)}(T) - F^{(j+1)}(T)] G^{(j+N)}(K)}{\mu \sum_{j=0}^{\infty} [F^{(j)}(T) - F^{(j+1)}(T)] \sum_{i=0}^{N+j-1} G^{(i)}(K)}. \quad (4)$$

When the unit is replaced at shock N,

$$C(N) \equiv \lim_{T \to 0} C(N,T) = \frac{c_F - (c_F - c_N) G^{(N)}(K)}{\mu \sum_{j=0}^{N-1} G^{(j)}(K)} \quad (N = 1, 2, \cdots). \quad (5)$$

When the unit is replaced at the first completion of shocks over time T is

$$C(T) \equiv C(1,T) = \frac{c_F - (c_F - c_N) \sum_{j=0}^{\infty} [F^{(j)}(T) - F^{(j+1)}(T)] G^{(j+1)}(K)}{\mu \sum_{j=0}^{\infty} [F^{(j)}(T) - F^{(j+1)}(T)] \sum_{i=0}^{j} G^{(i)}(K)}. \quad (6)$$

3. OPTIMAL REPLACEMENT POLICIES

When $F(t) = 1 - e^{-\lambda t}$ and $Q(N) \equiv [G^{(N)}(K) - G^{(N+1)}(K)]/G^{(N)}(K)$ increases strictly with N to 1, we derive optimal policies which minimize the expected cost rates. In this case, the expected cost rate in (4) is

$$\frac{C(N,T)}{\lambda} = \frac{c_F - (c_F - c_N) \sum_{j=0}^{\infty} [(\lambda T)^j / j!] e^{-\lambda T} G^{(j+N)}(K)}{\sum_{j=0}^{\infty} [(\lambda T)^j / j!] e^{-\lambda T} \sum_{i=0}^{N+j-1} G^{(i)}(K)}. \quad (7)$$

3.1 Optimal N^*

We find optimal N^* to minimizes $C(N)$ in (5). Forming the inequality $C(N+1) - C(N) \geq 0$

$$Q(N) \sum_{j=0}^{N-1} G^{(j)}(K) + G^{(N)}(K) \geq \frac{c_F}{c_F - c_N}, \quad (8)$$

where

$$Q(N,T) \equiv \frac{\sum_{j=0}^{\infty} [(\lambda T)^j / j!] [G^{(j+N)}(K) - G^{(j+N+1)}(K)]}{\sum_{j=0}^{\infty} [(\lambda T)^j / j!] G^{(j+N)}(K)},$$

$$Q(N) \equiv Q(N,0) = \frac{G^{(N)}(K) - G^{(N+1)}(K)}{G^{(N)}(K)},$$

$$Q(T) \equiv Q(1,T) = \frac{\sum_{j=0}^{\infty} [(\lambda T)^j / j!] [G^{(j+1)}(K) - G^{(j+2)}(K)]}{\sum_{j=0}^{\infty} [(\lambda T)^j / j!] G^{(j+1)}(K)}. \quad (9)$$

Note that $Q(N,T)$ increases strictly with N from $Q(T)$ to 1, and increases strictly with T from $Q(N)$ to 1 (Appendix). Thus, because the left-hand side of (8) increases strictly with N to $1 + M(K)$, where $M(K) \equiv \sum_{j=1}^{\infty} G^{(j)}(K)$. Therefore, if $M(K) > c_N/(c_F - c_N)$, then there exists a finite and unique minimum N^* $(1 \leq N^* < \infty)$ which satisfies (8).

3.2 Optimal T^*

We find optimal T^* to minimize $C(T)$ in (6). Differentiating $C(T)$ with respect to T and setting it equal to zero,

$$Q(T)\sum_{j=0}^{\infty}\frac{(\lambda T)^j}{j!}e^{-\lambda T}\sum_{i=0}^{j}G^{(i)}(K) + \sum_{j=0}^{\infty}\frac{(\lambda T)^j}{j!}e^{-\lambda T}G^{(j+1)}(K) = \frac{c_F}{c_F - c_N}, \tag{10}$$

whose left-hand side increases strictly with T to $1+M(K)$. Thus, if $M(K) > c_N/(c_F - c_N)$, then there exists a finite and unique T^* $(0 \le T^* \le \infty)$ which satisfies (10).

3.3 Optimal N_0^* and T_0^*

When $F(t) = 1 - e^{-\lambda t}$, $Q(N)$ increases strictly with N to 1 and $M(K) > c_N/(c_F - c_N)$, we find optimal N_0^* and T_0^* which minimize $C(N,T)$ in (7).

First, we find optimal N_0^* to minimizes $C(N,T)$ for fixed T $(0 \le T < \infty)$. Forming $C(N+1,T) - C(N,T) \ge 0$,

$$Q(N,T)\sum_{j=0}^{\infty}\frac{(\lambda T)^j}{j!}e^{-\lambda T}\sum_{i=0}^{N+j-1}G^{(i)}(K) + \sum_{j=0}^{\infty}\frac{(\lambda T)^j}{j!}e^{-\lambda T}G^{(j+N)}(K) \ge \frac{c_F}{c_F - c_N}, \tag{11}$$

whose left-hand side increases strictly with N to $1+M(K)$. Thus, if $M(K) > c_N/(c_F - c_N)$, then there exists a finite and unique minimum N_0^* $(1 \le N_0^* < \infty)$ which satisfies (11).

Letting $L(N,T)$ be the left-hand side of (11), $L(N,T)$ increases strictly with T from

$$L(N,0) = Q(N)\sum_{j=0}^{N-1}G^{(j)}(K) + G^{(N)}(K),$$

which agrees with (8). Thus, N_0^* decreases with T from N^* given in (8), and $1 \le N_0^* < N^*$. In addition, because $L(1,T)$ agrees with the left-hand side of (10), if $T \ge T^*$ given in (10), then $N_0^* = 1$, and conversely, if $T < T^*$ then $N_0^* \ge 2$.

Next, we find optimal T_0^* to minimize $C(N,T)$ for fixed N $(1 \le N < \infty)$. Differentiating $C(N,T)$ with respect to T and setting it equal to zero,

$$Q(N,T)\sum_{j=0}^{\infty}\frac{(\lambda T)^j}{j!}e^{-\lambda T}\sum_{i=0}^{N+j-1}G^{(i)}(K) + \sum_{j=0}^{\infty}\frac{(\lambda T)^j}{j!}e^{-\lambda T}G^{(j+N)}(K) = \frac{c_F}{c_F - c_N}, \tag{12}$$

Table 1: Optimal N_0^* when $\omega K = 10$.

c_F/c_N	λT						
	0	1	2	3	4	5	10
5	6	5	4	3	2	1	1
10	5	4	3	2	1	1	1
20	4	3	2	1	1	1	1
30	4	3	2	1	1	1	1
40	4	2	1	1	1	1	1
50	4	2	1	1	1	1	1

N^*

Table 2: Optimal N_0^* when $\omega K = 20$.

c_F/c_N	λT						
	0	1	2	3	4	5	10
5	13	12	11	10	9	8	2
10	12	10	9	8	7	6	1
20	10	9	8	7	6	5	1
30	10	9	7	6	5	4	1
40	10	8	7	6	5	3	1
50	9	8	7	6	4	3	1

N^*

whose left-hand side agrees with $L(N,T)$ and increases strictly with T from $L(N,0)$ given in (8) to $L(N,\infty) \ge M(K)$. Thus, because

$$L(N^*,T) \ge L(N^*,0) \ge \frac{c_F}{c_F - c_N},$$

$T_0^* = 0$, i.e., optimal policy which minimizes $C(N,T)$ is $N_0^* = N^*$ and $T_0^* = 0$. Therefore, replacement with shock N is better than replacement overtime when both replacement costs are the same.

Furthermore, if $N \ge N^*$, then $T_0^* = 0$, and conversely, if $N \le N^* - 1$, then $L(N,0) < c_F/(c_F - c_N)$ and there exists a finite and unique T_0^* $(0 < T_0^* < \infty)$ which satisfies (12).

Table 3: Optimal N_0^* when $\lambda T = 5$.

c_F/c_N	ωK					
	5	10	15	20	25	30
5	1	3	6	10	13	17
10	1	2	5	8	12	15
20	1	1	4	7	10	14
30	1	1	3	6	10	13
40	1	1	3	6	9	13
50	1	1	3	6	9	12

Table 4: Optimal λT_0^* when $\omega K = 10$.

c_F/c_N	N					
	1	2	3	4	5	10
5	4.7	3.7	2.6	1.6	0.6	0
10	3.4	2.4	1.4	0.5	0	0
20	2.5	1.6	0.7	0	0	0
30	2.1	1.2	0.4	0	0	0
40	1.9	1.0	0.2	0	0	0
50	1.7	0.8	0.0	0	0	0

T^*

4. NUMERICAL EXAMPLES

We give numerical examples when $F(t) = 1 - e^{-\lambda t}$ and $G(x) = 1 - e^{-\omega x}$. Then, for $N = 1,2, \dots$,

$$Q(N) = \frac{(\omega x)^N/N!}{\sum_{j=N}^{\infty}[(\omega x)^j/j!]}$$

increases strictly with N from $\omega x/(e^{\omega x} - 1)$ to 1, and

$$Q(N,T) = \frac{\sum_{j=0}^{\infty}[(\lambda T)^j/j!][(\omega x)^{j+N}/(j+N)!]}{\sum_{j=0}^{\infty}[(\lambda T)^j/j!]\sum_{i=j+N}^{\infty}[(\omega x)^i/i!]}$$

increases strictly with N from $Q(T)$ to 1 and increases strictly with T from $Q(N)$ to 1.

Table 5: Optimal λT_0^* when $\omega K = 20$.

c_F/c_N	N					
	1	2	3	4	5	10
5	10.9	9.9	8.9	8.0	7.0	2.3
10	9.0	8.1	7.2	6.3	5.4	0.9
20	7.7	6.8	6.0	5.1	4.2	0
30	7.1	6.2	5.3	4.5	3.6	0
40	6.7	5.8	5.0	4.1	3.3	0
50	6.4	5.5	4.7	3.9	3.0	0

T^*

Table 1 presents optimal N_0^* when $\omega K = 10$ for c_F/c_N and λT. We can see that N_0^* decreases with c_F/c_N. This indicates that if replacement cost c_F of failure is large, then we should replace the unit early to avoid its failure. Furthermore, N_0^* decreases with λT. This indicates that if the number of shocks is large, then we should replace early. Note that N_0^* decreases strictly with λT from N^* for $\lambda T = 0$ to 1.

Table 2 presents optimal N_0^* when $\omega K = 20$ for c_F/c_N and λT. We can see the same tendency with Table 1, and N_0^* is large when ωK is large.

Table 3 presents optimal N_0^* when λT=5 for c_F/c_N and ωK. This indicates that we should replace the unit early when ωK is small, because ωK means the expected number of damage to failure and the unit fails with a small number of shocks.

Table 4 presents optimal λT_0^* when $\omega K = 10$ for c_F/c_N and N. We can see that λT_0^* decreases with c_F/c_N. This indicates that if replacement cost c_F of failure is large, then we should replace the unit early to avoid its failure. Note that λT_0^* decreases strictly with N from λT^* for $N = 1$ to 0.

Table 5 presents optimal λT_0^* when $\omega K = 20$ for c_F/c_N and N. We can see the same tendency with Table 4. Further, we can see that λT_0^* increases with ωK. This indicates that if the expected number ωK of damage to failure is small, then we should replace the unit early to avoid its failure.

CONCLUSONS

We have proposed an extended model of the replacement overtime policy in which the unit is replaced at N th completion of shocks over planned time T. Further, the units fails when the total damage exceeded a prespecified level K.We have obtained the expected cost rates, and discussed optimal T^* and N^* which minimize them. As a future work, we should modify this model more realistically for equipment management. For example, we could consider maintenance policies that the equipment is replaced at random time over planned time. Another example is that the number of shocks over time is given as random variable. These formulations and results would be applied to real systems such as management projects to develop information system effectively by suitable modifications.

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APPENDIX

When $Q(N)$ increases strictly with N to 1,

$$\frac{\sum_{j=0}^{\infty}[(\lambda T)^j/j!][G^{(j+N+1)}(K)-G^{(j+N+2)}(K)]}{\sum_{j=0}^{\infty}[(\lambda T)^j/j!]G^{(j+N+1)}(K)} - \frac{\sum_{j=0}^{\infty}[(\lambda T)^j/j!][G^{(j+N)}(K)-G^{(j+N+1)}(K)]}{\sum_{j=0}^{\infty}[(\lambda T)^j/j!]G^{(j+N)}(K)} > 0,$$

increases strictly with N from $Q(N)$ to 1 and increases strictly with T from $Q(N)$ to 1.

Proof. First, note that for any $N_1 > 0$,

$$\lim_{N\to\infty}\frac{\sum_{j=0}^{N_1}[(\lambda T)^j/j!][G^{(j+N)}(K)-G^{(j+N+1)}(K)]}{\sum_{j=0}^{N_1}[(\lambda T)^j/j!]G^{(j+N)}(K)} = \lim_{N\to\infty}\frac{G^{(N_1+N)}(K)-G^{(N_1+N+1)}(K)}{G^{N_1+N}(K)} = \lim_{N\to\infty} Q(N) = 1,$$

and

$$\lim_{T\to\infty}\frac{\sum_{j=0}^{N_1}[(\lambda T)^j/j!][G^{(j+N)}(K)-G^{(j+N+1)}(K)]}{\sum_{j=0}^{N_1}[(\lambda T)^j/j!]G^{(j+N)}(K)} = \frac{G^{(N_1+N)}(K)-G^{(N_1+N+1)}(K)}{G^{N_1+N}(K)} = Q(N_1+N),$$

which follows that $\lim_{N\to\infty}Q(N,T) = \lim_{T\to\infty}Q(N,T) = 1$ because N_1 is arbitrary.

Next, because $Q(N,T)$ is rewritten as

$$Q(N,T) = \frac{\sum_{j=N}^{\infty}[(\lambda T)^{j-N}/(j-N)!][G^{(j)}(K)-G^{(j+1)}(K)]}{\sum_{j=N}^{\infty}[(\lambda T)^{j-N}/(j-N)!]G^{(j)}(K)},$$

from $Q(N+1,T) - Q(N,T)$,

$$G^{(N)}(K)\sum_{j=N}^{\infty}\frac{(\lambda T)^{j-N}}{(j-N)!}[G^{(j)}(K)-G^{(j+1)}(K)] - [G^{(N)}(K)-G^{(N+1)}(K)]\sum_{j=N}^{\infty}\frac{(\lambda T)^{j-N}}{(j-N)!}G^{(j)}(K) = G^{(N)}(K)\sum_{j=N}^{\infty}\frac{(\lambda T)^{j-N}}{(j-N)!}G^{(j)}(K)[Q(j)-Q(N)] > 0,$$

which follows that $Q(N,T)$ increases strictly with N to 1.Differentiating $Q(N,T)$ with respect to T,

$$\frac{\sum_{j=0}^{\infty}[(\lambda T)^j/j!][G^{(j+N+1)}(K)-G^{(j+N+2)}(K)]}{\sum_{j=0}^{\infty}[(\lambda T)^j/j!]G^{(j+N+1)}(K)} - \frac{\sum_{j=0}^{\infty}[(\lambda T)^j/j!][G^{(j+N)}(K)-G^{(j+N+1)}(K)]}{\sum_{j=0}^{\infty}[(\lambda T)^j/j!]G^{(j+N)}(K)} > 0,$$

which follows that $Q(N,T)$ increases strictly with T to 1.