Asymmetric multi-fractality and market efficiency in stock indices of G-2 countries

Minhyuk Lee
Department of Industrial Engineering
Seoul National University, Seoul, Korea
Tel: (+82) 2-883-8336, Email: superou@snu.ac.kr

Jae Wook Song
Department of Industrial Engineering
Seoul National University, Seoul, Korea
Tel: (+82) 2-883-8336, Email: songjw87@snu.ac.kr

Ji Hwan Park
Department of Industrial Engineering
Seoul National University, Seoul, Korea
Tel: (+82) 2-883-8336, Email: park3018@snu.ac.kr

Woojin Chang †
Department of Industrial Engineering
Seoul National University, Seoul, Korea
Tel: (+82) 2-880-8335, Email: changw@snu.ac.kr

Abstract. The aim of this paper is to detect the asymmetric multi-fractality in stock indices of G-2 countries based on the asymmetric multi-fractal detrended fluctuation analysis (A-MFDFA). A-MFDFA is proposed to measure the scaling behavior of time series; it defines the state of mono- and multi-fractality which indicate the efficient and inefficient market, respectively. In addition to the conventional return-based criteria for market trend, we introduce the index-based model so that the stylized facts of asymmetric multi-fractality can be revealed. Based on two models, we provide the evidences of asymmetric multi-fractality and time-varying efficiency of each stock market with respect to different time window.

Keywords: Asymmetric multi-fractality; MFDFA; Market efficiency; Scaling behavior

1. INTRODUCTION

In previous literatures, the multi-fractal analysis has been applied to investigate the various stylized facts of the financial market including market inefficiency (Cajuerio and Tabak, 2004; Wang et al., 2010), risk evaluation (Lee et al., 2016), and crash prediction (Grech and Mazur, 2004). Specifically, the multi-fractal detrended fluctuation analysis (MF-DFA) is a typical approach to measure the long-range autocorrelations and multi-fractality of a time-series (Kantelhardt et al., 2002), a generalization of the detrended fluctuation analysis (DFA) (Peng et al., 1994). Many studies have analyzed the multi-fractal behaviors of the stock markets (Greene and Fielitz, 1977; Lee et al., 2006; Sun et al., 2001), but only few of them focus on the asymmetric multi-fractal scaling behavior.

The stock market is composed of two asymmetric market trends, known as the bullish and bearish markets, and they should be treated differently in analyzing the multi-fractal scaling behavior and asymmetric correlation. In general, the presence of asymmetric correlation can affect the portfolio diversification and risk management (Ang and Chen, 2002). Alvarez-Ramirez et al. (2009) introduce the asymmetric DFA (A-DFA). In extension to A-DFA, Cao et al. (2013) propose the asymmetric multi-fractal detrended fluctuation analysis (A-MFDFA) method to examine the asymmetric multi-fractal scaling behaviors of uptrends and downtrends. Interestingly, Cao et al. (2013)

demonstrate the distinct scaling properties in two different market trends where the up- and down-trends are distinguished based on the linear regression of return dynamics. In addition to the return dynamics, we suggest to include the index dynamics as a new criterion for separating the market trends, which is more intuitive measure of ups and downs.

In this paper, we employ A-MFDFA method for analyzing the stock indices of the United States and China (a.k.a. G-2 countries) so that the existence of asymmetric multi-fractal scaling behavior can be observed. Specifically, A-MFDFA is utilized into two models in accordance with the separation criterion of asymmetric trends: one is return-based and the other is index-based model. Based on two models, we discuss the empirical difference of two models, features of scaling behavior, and the degree of market efficiency in the G-2 stock indices. Furthermore, we investigate the scaling asymmetries and the time-varying feature of asymmetric multi-fractality.

2. A-MFDFA METHODOLOGY

Let $\{x(t)\}$ be the time series $t = 1, 2, \dots, N$ where N denotes the length of the series, then A-MFDFA can be defined in the following steps:

***Step 1**: Construct the profile.*

$$y(j) = \sum_{t=1}^{j} (x(t) - \bar{x}), \quad j = 1, 2, \dots, N \tag{1}$$

where $\bar{x} = \frac{1}{N}\sum_{t=1}^{N} x(t)$.

***Step 2**: Divide time-series to non-overlapping sub-time*

The time-series and its profile are divided into $N_n = int(N/n)$ non-overlapping sub-time series of equal length n. The length N may not be a multiple of time-scale n, so a short part of the profile will remain in most cases. We repeat this procedure from the other end of the profile, which leads to obtain $2N_n$ sub-time series. Suppose $S_j = \{s_{j,k}, k = 1,2, \dots, n\}$ be the j^{th} sub-time series of length n and $Y_j = \{y_{j,k}, k = 1,2, \dots, n\}$ be the profile time series in j^{th} time interval for $j = 1,2, \dots, 2N_n$. Then, we have

$$s_{j,k} = x((j-1)n + k), \qquad y_{j,k} = y((j-1)n + k) \tag{2}$$

for $j = 1, 2, \dots, N_n$ and

$$s_{j,k} = x(N - (j - N_n)n + k)$$
$$y_{j,k} = y(N - (j - N_n)n + k) \tag{3}$$

for $j = N_n + 1, \dots, 2N_n$. Note that $5 \le n \le N/6$ is selected.

***Step 3**: Construct the fluctuation function*

For each S_j and Y_j, we compute the local least-squares fits $L_{S_j}(k) = a_{s_j} + b_{s_j}k$ and $L_{Y_j}(k) = a_{Y_j} + b_{Y_j}k$, where k presents the horizontal coordinate. The slope of $L_{S_j}(k)$, b_{S_j}, is used to discriminate whether the trend of S_j is positive or negative. L_{Y_j} is used to detrend the integrated time series Y_j. So, we define the fluctuation functions such that,

$$F_j(n) = \frac{1}{n}\sum_{k=1}^{n} \left(y_{j,k} - L_{Y_j}(k)\right)^2 \tag{4}$$

for $j = 1, 2, \dots, 2N_n$.

***Step 4-1**: Divide trend using return dynamics*

Suppose the time series $x(t)$ has piecewise positive and negative linear trends. Then, the fluctuation functions are considered to assess asymmetric cross-correlation scaling properties based on the sign of the slope, b_{S_j}. It defines that $b_{S_j} > 0$ ($b_{S_j} < 0$) is true if the time series $x(t)$ has a positive (negative) trend in the S_j. This model is a conventional approach called Return-based A-MFDFA.

***Step 4-2**: Divide trend using index dynamics*

Let $I(t) = I(t-1)\,exp(x(t))$ for $t = 1,2, \dots, N$, where $I(0) = 1$, then $\{I(t)\}$ is the indexing proxy of return time series. Similar to *Step 2*, suppose $G_j = \{g_{j,k}, k = 1,2, \dots, n\}$ be the j^{th} sub-time series of length n. Then, we have

$$g_{j,k} = I((j-1)n + k), \qquad j = 1,2, \dots, N_n$$
$$g_{j,k} = I(N - (j - N_n)n + k)), \qquad j = N_n + 1, \dots, 2N_n \tag{5}$$

for $j = N_n + 1, \dots, 2N_n$. Note that $5 \le n \le N/6$ is selected.

Now, we can compute the local least-squares fits $L_{G_j}(k) = a_{G_j} + b_{G_j}k$, where k presents the horizontal coordinate. The slope of $L_{G_j}(k)$, b_{G_j}, is used to discriminate whether the trend of G_j is positive or negative. It defines that $b_{G_j} > 0$ ($b_{G_j} < 0$) is true if the time series $I(t)$ has a positive (negative) trend in the G_j. We call this model as Index-based A-MFDFA.

Step 5: Construct q-order average fluctuation functions

Then, the directional q-order average fluctuation functions of return-based model can be computed by,

$$F_q^+(n) = \left(\frac{1}{M^+} \sum_{j=1}^{2N_n} \frac{sign\left(b_{S_j}\right) + 1}{2} \left[F_j(n)\right]^{\frac{q}{2}} \right)^{\frac{1}{q}}$$
$$F_q^-(n) = \left(\frac{1}{M^-} \sum_{j=1}^{2N_n} \frac{-\left[sign\left(b_{S_j}\right) - 1\right]}{2} \left[F_j(n)\right]^{\frac{q}{2}} \right)^{\frac{1}{q}} \quad (6)$$

where $M^+ = \sum_{j=1}^{2N_n}\left[sign\left(b_{S_j}\right) + 1\right]/2$ and $M^- = \sum_{j=1}^{2N_n} - \left[sign\left(b_{S_j}\right) - 1\right]/2$, which denotes the number of sub-time series with positive and negative trends, respectively.

The average fluctuation function also can be computed as,

$$F_q(n) = \left(\frac{1}{2N_n} \sum_{j=1}^{2N_n} \left[F_j(n)\right]^{\frac{q}{2}} \right)^{\frac{1}{q}} \quad (7)$$

Furthermore, the index-based average fluctuation function can be computed by altering b_{S_j} with b_{G_j}.

Step 6: Calculating the generalized Hurst exponent

When the power-law relation exists, $H(q)$, $H^+(q)$, and $H^-(q)$ denote the overall, upward, and downward scaling exponents, respectively. Specifically, the scaling satisfies,

$$F_q(n) \sim n^{H(q)}; \qquad F_q^+(n) \sim n^{H^+(q)}; \qquad F_q^-(n) \sim n^{H^-(} \quad (8)$$

These relations then can be changed into,

$$\log F_q(n) = H(q) \log n + \log A_1$$
$$\log F_q^+(n) = H^+(q) \log n + \log A_2 \quad (9)$$
$$\log F_q^-(n) = H^-(q) \log n + \log A_3$$

$H(q)$ is the generalized Hurst exponent, which indicates the multi-fractal time-series if $H(q)$ depends on q. Otherwise, the time-series are mono-fractal. In general, following relation exists.

$$H(2) \begin{cases} > 0.5 & \text{: Persistent} \\ < 0.5 & \text{: Anti-persistent} \\ = 0.5 & \text{: Random walk} \end{cases}$$

Analogous to $H(q)$, the uptrend (downtrend) time-series are multi-fractal if the time-series shows positive (negative) trend and $H^+(q)$ ($H^-(q)$) depends on q. Otherwise, the uptrend (downtrend) time-series are mono-fractal.

3. DATA

This paper uses the daily closing prices of the U.S. (DJIA and NASDAQ) and China (SSCI and SZCI) indices. Note that DJIA, NASDAQ, SSCI, SZCI are abbreviations for Dow Jones Industrial Average Index, National Association of Securities Dealers Automated Quotations Composite Index, Shanghai Stock Exchange Composite Index, and the Shenzhen Component Index, respectively. The experimental period of time-series is considered from 1991-01-01 to 2015-12-31 for most indices except SZCI which started to operate in 1991-04-03. Then, we transform the price-series to the logarithmic return-series, $r_t = log(P_t) - log(P_{t-1})$, where P_t is the closing price index at time t. Finally, the sample sizes of NJIA, NASDAQ, SSCI, and SZCI are 6290, 6293, 6122 and 6013 trading dates, respectively. Note that the price data can be obtained from any open platform such as Yahoo and Google finance.

4. RESULTS

4.1 Asymmetric fluctuation functions and their dynamics

In this section, two models of A-MFDFA are applied to investigate the aspects of fluctuation functions with respect to different trends. As stated in Chapter 3, the experiment targets are the stock indices in the U.S. and China. In Fig.1 to 4, the results of A-MFDFA when $q = 2$ are illustrated for each indices. Specifically, Figures on the left demonstrate the $\log_2(F_2(n))$ vs. $\log_2(n)$ result of return-based model, whereas Figures on the right demonstrate that of index-based model. Note that blue, red, and yellow dots represent the overall, upwards, and downwards, respectively. It is well-known stylized fact that $\log_2(F_2(n))$ vs. $\log_2(n)$ possesses a power-law dependency where the straight dotted line indicates a decent power-law fit. In general, the asymmetry in fluctuation functions is discovered within a single unit of time-scale where the distinctions between the values of uptrend and downtrend are observed through most of time-scale. Besides, the dynamics of fluctuation functions exhibit the symmetric evolution in accordance with the time-scale increment. In addition, the newly-suggested approach of index-based model clearly distinguishes the

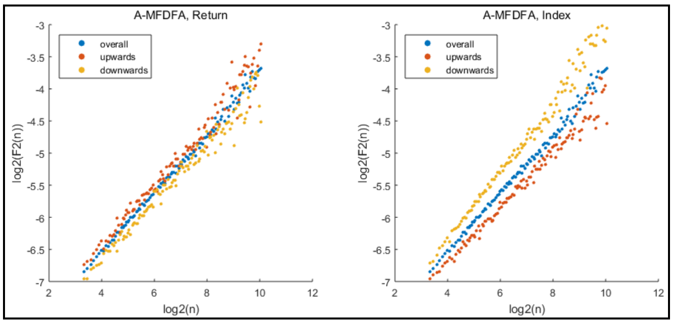

Figure 1: plots of $\log_2(F_2(n))$ vs. $\log_2(n)$ for DJIA

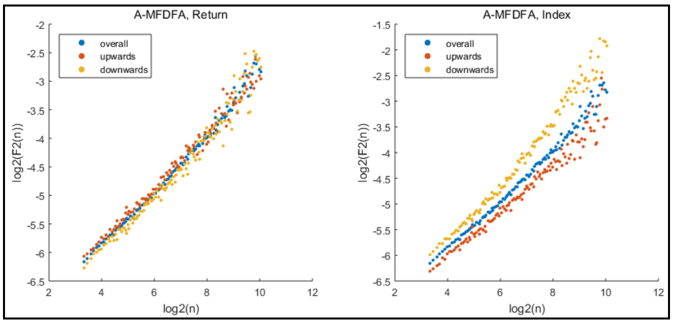

Figure 2: plots of $\log_2(F_2(n))$ vs. $\log_2(n)$ for NASDAQ

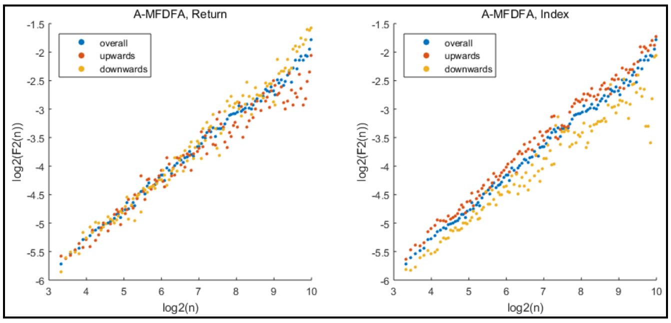

Figure 3: plots of $\log_2(F_2(n))$ vs. $\log_2(n)$ for SSCI

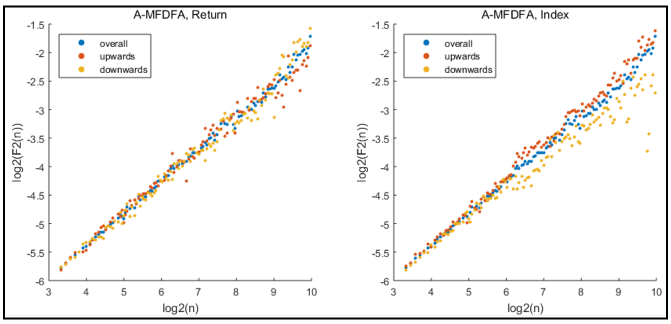

Figure 4: plots of $\log_2(F_2(n))$ vs. $\log_2(n)$ for SZCI

straight trend of upwards and downwards pivoting on the overall dots, whereas the conventional approach of return-based model shows the dots with scatter distribution without a straight trend. Hence, the results suggest that the index-based model provides more robust criterion of detecting the power-law scaling property. In other words, the index-based model performs better clustering of two different trends.

Furthermore, the fluctuation functions of trends show the reverse order of their values between the return- and index-based models. In cases of DJIA and NASDAQ, the descending order of return-based model is upward, overall, and downward, whereas that of index-based model is downward, overall, and upward. However, SSCI and SZCI show the completely opposite orders of each model in comparison to the U.S. indices.

4.2 Time-varying multi-fractality and market efficiency

Wang et al., (2010) suggest that the degree of multi-fractality can measure the market efficiency. Thus, the degree of multi-fractality, $\Delta H = max(H(q)) - min(H(q))$, can be a proxy of the degree of market efficiency. Note that $\Delta H = 0$ indicates the perfectly efficient market. In contrast, the market is inefficient if ΔH is deviated from 0.

Table 1 lists the values of ΔH in different indices for total period of experiment, which is equivalent to 25 years. Note that ΔH^+ and ΔH^- are computed based on $\Delta H^{\pm} = max\left(H^{\pm}(q)\right) - min\left(H^{\pm}(q)\right)$. Based on the results, the most efficient market is discovered to be NASDAQ for both models. Then, the efficiency is generally ordered as DJIA, SZCI, and SSCI except the case of downward trend in index-based model where SSCI is more efficient than SZCI. Hence, it can be concluded that the U.S. indices are more efficient than the Chinese indices.

Table 2 shows the result of cross-sectional multi-fractality of each index based on 5 years of time window. The result is expected to reveal the time-varying property of multi-fractality.

For the U.S. stock indices, ΔH differs with respect to different time window, which can be explained by the macro or financial crisis periods. Note that the market is known to be inefficient during the crisis period. In case of DJIA in return-based model, ΔH^- is substantially large during the time window of 2006-2010 where the outbreak of sub-prime mortgage crisis destroyed the U.S. financial system. In case of NASDAQ in index-based model, ΔH^- is also distinguishably large during the time windows of 1996-2000 and 2011-2015 where the Dot-com bubble and European debt crisis were presented.

In contrast to the case of U.S. stock indices, ΔH of Chinese stock indices are gradually reduced with respect to

Table 1: ΔH of each time series from 1991 to 2015

Trend	DJIA		NASDAQ		SSCI		SZCI	
	Return	Index	Return	Index	Return	Index	Return	Index
Overall	0.1330		0.0467		0.6718		0.3754	
Upward	0.1432	0.1983	0.1193	0.1144	0.7115	0.8036	0.3972	0.2402
Downward	0.1623	0.1682	0.0459	0.2680	0.5754	0.3261	0.3189	0.4741

Table 2: ΔH of cross-sectional multi-fractality based on 5 years of time window

Period	Trend	DJIA		NASDAQ		SSCI		SZCI	
		Return	Index	Return	Index	Return	Index	Return	Index
1991 – 1995	Overall	0.1118		0.0602		0.9257		0.8433	
	Upward	0.1817	0.1198	0.1531	0.0554	0.7908	1.0900	0.9473	0.4426
	Downward	0.0690	0.2019	0.0606	0.1510	0.9276	0.2972	0.7575	0.8692
1996 - 2000	Overall	0.1952		0.1744		0.4479		0.4667	
	Upward	0.2046	0.1819	0.2434	0.1544	0.4260	0.4128	0.4808	0.3731
	Downward	0.2083	0.2422	0.1620	0.3486	0.4406	0.5521	0.4595	0.5684
2001 – 2005	Overall	0.1687		0.1245		0.2584		0.3101	
	Upward	0.2403	0.2258	0.1720	0.3042	0.3059	0.3752	0.3024	0.3913
	Downward	0.1689	0.2718	0.1878	0.2627	0.2425	0.2013	0.3248	0.3011
2006 – 2010	Overall	0.3022		0.1704		0.2725		0.2583	
	Upward	0.2571	0.3674	0.2289	0.2713	0.2986	0.3016	0.2824	0.2903
	Downward	0.4087	0.2610	0.2135	0.1724	0.2751	0.1478	0.2678	0.1866
2011 - 2015	Overall	0.2770		0.2564		0.1280		0.0370	
	Upward	0.2705	0.2841	0.2651	0.2179	0.2979	0.0862	0.2428	0.0849
	Downward	0.2914	0.3017	0.2813	0.3592	0.0824	0.2412	0.0710	0.2408

time. The reason behind such phenomenon can be explained by the financial reform of Chinese stock market on 1996-12-16 where $\pm 10\%$ of the price limit is established to prevent the speculative investment. In succession to the reform, ΔH of the Chinese indices in the time window of 1996-2000 shows dramatic reduction in comparison to that of 1991-1995. It also indicates that the reform set the market to be more efficient.

5. CONCLUSION

In this paper, we apply the return- and index-based models of A-MFDFA to investigate the asymmetric multi-fractal scaling behavior of the stock indices in G-2 countries.

At first, the asymmetry in fluctuation functions is detected through most of time-scale for all two models. In addition, the dynamics of fluctuation functions show the symmetric evolution. However, the index-based model separates the asymmetric scaling behavior more clearly than return-based model. Specifically, the phenomenon insists that the index-based model is more suitable in detecting the power-law dependency than the return-based one. Furthermore, the values of fluctuation functions for each trend exhibit the reversed order between return- and index-based models.

Secondly, we explore the market efficiency of the U.S. and China stock markets using the degree of multi-fractality. For the total period, NASDAQ is most efficient market and DJIA, SZCI and SSCI show their efficiency in descending order. Additionally, the results of time-varying multi-fractality confirm the different market efficiency with respect to specific time window. In the U.S. stock market, ΔH^- of return-based model indicates the inefficient market during sub-prime mortgage crisis, whereas that of index-based model depicts the inefficient market during Dot-com bubble and European-debt crisis. Nevertheless, the Chinese market exhibits the gradually decreasing $\Delta H, \Delta H^+$ and ΔH^-, which suggests that the market is becoming more efficient. Since the dramatic improvement of market efficiency is found in 1996-2000, we presume that the financial reform of China is the behind reason.

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