Approximations for Reliability of The Connected-(1,2)-or-(2,1)-out-of-(m,n):F Lattice System

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Abstract. The connected-(1,2)-or-(2,1)-out-of-(m,n):F lattice system consists $m \times n$ components, and it is failed if and only if there exists at least one of a (1,2)-matrix and a (2,1)-matrix. This system can be applied to various systems in practice e.g., the supervision system. Several calculation methods have been proposed to obtain the exact reliability of the connected-(1,2)-or-(2,1)-out-of-(m,n):F lattice system. However, these methods have the weak point that their calculating times increase with system size. Hence, the evaluation of approximations for the system reliability, which can be calculate in a practical calculating time, is important. We derive new upper and lower bounds for evaluating the reliability of a connected-(1,2)-or-(2,1)-out-of-(m,n):F lattice system for the i.i.d. case. Then, we show some numerical examples in order to confirm the efficiency of the proposed approximate values.

Keywords: connected-(1,2)-or-(2,1)-out-of-(m,n):F lattice system, upper bound, lower bound

1. INTRODUCTION

A linear consecutive-k-out-of-n:F system consists of n components arranged in a line. The system fails if and only if at least k consecutive components fail. It was firstly introduced by Kontolen (1980) and has been extensively studied by Chang *et al*. (1995), Kuo and Zuo (2002) and Chao *et al.* (1995) etc. Furthermore, in recent years, the studies have been reviewed by Eryilmaz (2010) and Triantafyllou (2015). A linear connected-X-out-of-(m,n):F lattice system is a two-dimensional version of the linear consecutive-k-out-of-n:F system (Boehme *et al.* (1992)). The system fails if and only if at least one subset of connected failed components occurs which includes failed components connected in the meaning of "connected-X". For example, a connected-(1,2)-or-(2,1)-out-of-(m,n):F lattice system consists of $m \times n$ components arranged in m rows by n columns. It fails if and only if there exists at least one of a (1,2)-matrix and a (2,1)-matrix (see Fig. 1). This system can be applied to various systems in practice e.g., the supervision system. Several calculation methods have been proposed to obtain the exact reliability of the connected-(1,2)-or-(2,1)-out-of-(m,n):F lattice system (Yamamoto *et al.* (2007)). However, these methods have the weak point that their calculating times increase with system size. Hence, the evaluation of approximations for the system reliability, which can be calculate in a practical calculating time, is important.

Many studies have given upper and lower bounds for a two-dimensional system. Yamamoto and Miyakawa (1995), Makinowski and Preuss (1996) and Boutsikas and Koutras (2000) proposed bounds for a connected-(r,s)-out-of-(m,n):F lattice system in which the components are ordered like the elements of an (m,n)-matrix so that the system fails if at least one connected (r,s)-submatrix fails. However, as far as we know, the only literature that can calculate upper and lower bounds for a connected-(1,2)-or-(2,1)-out-of-(m,n):F lattice system is Yamamoto (1996). Note that, in nature, Yamamoto (1996) gives upper and lower bounds for the reliability of the connected-X-out-of-(m,n):F lattice system, which is more general system than a connected-(1,2)-or-(2,1)-out-of-(m,n):F lattice system.

The aim of this paper is to derive new upper and lower bounds for evaluating the reliability of a connected-(1,2)-or-(2,1)-out-of-(m,n):F lattice system for the i.i.d. case. Additionally, we perform numerical examples in order to compare with the existing bounds (Yamamoto (1996)). In Section 2, we clarify the system size whose reliability can be obtained by explicit solution technique and present the existing upper and lower bounds. In Section 3, we propose new formulas for upper and lower bounds. In Section 4, we show numerical examples. In Section 5, we discuss our conclusions and future work.

2. PREVIOUS STUDIES

In this section, we clarify the system size whose reliability can be obtained by explicit solution technique. Furthermore, we apply Yamamoto (1996)'s bounds to the connected-(1,2)-or-(2,1)-out-of-(m,n):F lattice system and confirm the efficacy.

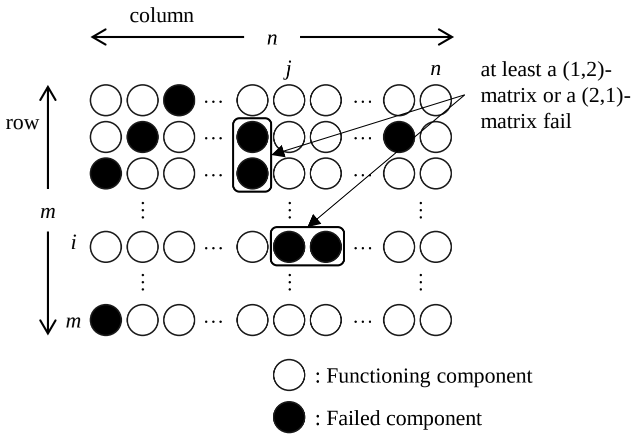

Figure 1: Example of a connected-(1,2)-or-(2,1)-out-of-(m,n):F lattice system failure.

In this paper, we denote several notations. Let p be the reliability of each component and q be the failure probability of each component, where $p+q=1$. We assume that all component are mutually s-independent and have the same component reliability. For $i=1,2,\ldots,m$ and $j=1,2,\ldots,n$, we define the indicator variable x_{ij} by

$$x_{ij} = \begin{cases} 1, & \text{if the component}(i, j) \text{ is functioning}, \\ 0, & \text{if the component}(i, j) \text{ has failed}, \end{cases} \tag{1}$$

where the component (i, j) means the component located on i th row and j th column. For $j=1,2,\ldots,n$, let the vector $\mathbf{x}_j = (x_{1j}, x_{2j}, \ldots, x_{mj})$ be the state vector.

2.1 Explicit Solution

If the system size for evaluation is not large, by using explicit solutions, little calculating time is required. In this subsection, we present the calculation method based on Markov chain proposed by Nakamura *et al.* (2016). Note that, in nature, the calculation method proposed by Nakamura *et al.* (2016) evaluate the reliability of a toroidal connected-(1,2)-or-(2,1)-out-of-(m,n):F lattice system. Here we apply it to linear connected-(1,2)-or-(2,1)-out-of-(m,n):F lattice system and investigate the calculating time. Hence, we rewrite the calculation method in order to evaluate the reliability of the linear connected-(1,2)-or-(2,1)-out-of-(m,n):F lattice system.

Next, we show a rewritten calculation method. It is employed to calculate the proposed upper and lower bounds in Section 3. For $a=1,2,\ldots,2^m$ and $b=1,2,\ldots,2^m$, the transition probability of the Markov chain is given by

$$m_{ba} = p^{m-\alpha} \times q^{\alpha} \times \phi(\mathbf{x}_j, \mathbf{x}_{j-1}), \tag{2}$$

where $\alpha = \sum_{k=1}^{m} x_{kj}$, and

$$\begin{aligned}\phi(\mathbf{x}_j, \mathbf{x}_{j-1}) &= \prod_{i=1}^{m} (1 - x_{ij} \cdot x_{i,j-1}) \\ &\times \prod_{i=1}^{m} (1 - x_{ij} \cdot x_{i-1,j})(1 - x_{i,j-1} \cdot x_{i-1,j-1}).\end{aligned} \tag{3}$$

$\phi(\mathbf{x}_j, \mathbf{x}_{j-1})$ means an indicator function, which takes 1 when the system failure pattern does not occur in either j th column, $j-1$th column or the two columns straddled j th column and $j-1$ th column. From Eqs. (2) and (3), the transition probability matrix is denoted as follows:

$$\mathbf{M} = (m_{ba}) \in \mathbf{R}^{2^m \times 2^m} . \tag{4}$$

Then, the reliability of the connected-(1,2)-or-(2,1)-out-of-(m,n):F lattice system is

$$R(m,n) = \mathbf{u}^T (\mathbf{M}^n) \boldsymbol{\pi}_0, \tag{5}$$

where $\boldsymbol{\pi}_0 = (1,0,\cdots,0)^T \in \mathbf{R}^{2^m \times 1}$ is the initial probability vector and $\mathbf{u} = (1,1,\cdots,1)^T \in \mathbf{R}^{2^m \times 1}$ is the probability vector, which sums up whole reliabilities of functioning systems.

We present numerical examples of the explicit solutions (Eq. (5)). The calculating time is estimated with a Windows 10 Intel Core i5 3.20GHz 4GB, and MATLAB R2016a. The results for systems fixed m are shown in Table1, and the results for systems fixed n are shown in Table2. Table1 shows that when the number of rows is large, little calculating time is required. On the other hand, Table 2 shows that when the number of column is large, much calculating time is required for exact system reliability. However, since parameter m and parameter n are interchangeable, we can also calculate the reliability of the system with large m. Thus, this method based on Markov chain is effective when either large m or large n. However, this method has the weak point that its calculating time increases in the case of that both parameter m and parameter n are large. The recursive equation proposed by Yamamoto *et al.* (2007) has similar results to the method based on Markov chain proposed by Nakamura *et al*. (2016). Although some explicit solutions have been proposed, their calculating times increase as the system size become large. Hence, the evaluation of upper and lower bounds for the system reliability, which can be calculate in a practical calculating time, is important. Therefore, we consider upper and lower bounds in this paper.

2.2 Upper and Lower Bounds

In this subsection, we present the upper and lower bounds proposed by Yamamoto (1996). Note that, in nature, Yamamoto (1996) gives upper and lower bounds for the reliability of the connected-X-out-of-(m,n):F lattice system.

In this subsection, we calculate the upper and lower bounds for the reliability of the connected-(1,2)-or-(2,1)-out-of-(m,n):F lattice system. We present upper and lower bounds for the reliability of connected-X-out-of-(m,n):F lattice system. For $\theta = 1,2...,h$, let X_θ be one of the failure patterns, and X means X_1 or X_2 or…or X_h. Without loss of generality, we assume that a failure pattern X_θ does not include others for all θ. For $\theta = 1,2...,h$, let r_θ be the number of rows of the smallest rectangle which encloses the failure pattern X_θ, and let s_θ be the number of columns of the smallest rectangle. For $i = 1,2,\ldots,m-r_{\min}+1$ and $j = 1,2,\ldots,n-s_{\min}+1$, τ is written as

$$\tau = i + (j-1)(m - r_{\min} + 1), \tag{6}$$

and $\gamma_{\theta\tau}$ is defined by random variable that takes 0 when the failure pattern X_θ in the rectangle with component (i,j) as its upper left corner occurs, and 1 otherwise. Here τ performs one-to-one correspondence to (i,j) and we need to

Table 1: Calculating time when m = 1000.

m	8	⋯	12	13	14	15	16	17	18
Calculating time (sec.)	0.041	⋯	0.271	0.663	1.867	5.259	15.143	44.888	148.566

Table 2: Calculating time when n = 12.

n	1000	4000	16000	64000	256000	1024000	4096000
Calculating time (sec.)	1.855	1.902	1.943	2.05	2.072	2.129	2.196

express $\tau(i, j)$ in nature. However, for the sake of simplicity expression, we express τ hereinafter. Let B_τ be the event that all component lying adjacent to the $r_{\max} \times s_{\max}$ rectangle with component (i, j) as its upper left corner. Namely,

$$B_\tau = \left\{ \left(\prod_{\mu=j}^{j+s_{\max}-1} x_{i-1,\mu} \right) \left(\prod_{\nu=j}^{i+r_{\max}-1} x_{\nu, j-1} \right) = 1 \right\}, \tag{7}$$

where $\tau = i + (j-1)(m - r_{\min} + 1)$ for $i = 1,2,\ldots,m - r_{\min} + 1$ and $j = 1,2,\ldots,n - s_{\min} + 1$. By using the above notations, the lower bound LB_Y and the upper bound UB_Y for the connected-X-out-of-(m,n):F lattice system are obtained as follows:

$$LB_Y = \prod_{i=1}^{m-r_{\min}+1} \prod_{j=1}^{n-s_{\min}+1} \prod_{\theta=1}^{k} [1 - \Pr\{\gamma_{\theta\tau} = 0\}], \tag{8}$$

$$UB_Y = \prod_{i=1}^{m-r_{\min}+1} \prod_{j=1}^{n-s_{\min}+1} [1 - \Pr\{B_\tau\} \Pr\{A_\tau\}], \tag{9}$$

where $A_\tau = \bigcup_{\theta=1}^{k} \{\gamma_{\theta\tau} = 0\}$.

B_τ is defined as Eq. (7) in order to obtain the upper bounds for the general connected-X-out-of-(m,n):F lattice system. When the failure patterns X is given, if we select the event B_τ so that the event A_τ and the event $B_\tau \cap (A_1^c \cap A_2^c \cap \cdots \cap A_{\tau-1}^c)$ are independent and the number of components composed of the event B_τ is smallest, we can obtain the better upper bound.

The lower and upper bounds of Eqs. (8) and (9) are obtained by ignoring the dependency of the minimal cut sets. Hence, since the connected-(1,2)-or-(2,1)-out-of-(m,n):F lattice system has more cut sets than any other 2-dimensional system relatively, we estimate that the errors between the exact system reliability and bounds tend to be large.

3. PROPOSAL OF UPPER AND LOWER BOUNDS

In this section, for the connected-(1,2)-or-(2,1)-out-of-(m,n):F lattice system whose bounds tend to increase the error, by extending ideas of Yamamoto and Miyakawa (1995) or Malinowski and Preuss (1996), we propose new formulas for upper and lower bounds. The numerical examples in the subsection 2.1 shows the calculation method of Eq. (5) is not effective in the case that both parameter m and parameter n are large. In other words, we can obtain the reliability of the connected-(1,2)-or-(2,1)-out-of-(m,n):F lattice system when either large m or large n in an efficient way. In this paper, we regard the system which can be obtained in a short time as a part of the connected-(1,2)-or-(2,1)-out-of-(m,n):F lattice system and calculate the bounds by using it. If we can reduce the number of times that we ignore the dependency of the minimal cut sets, we obtain the ls which are more useful than the exacting bounds. Hence, we propose new formulas for upper and lower bounds.

First, we define the following notations. Let $R_{\text{sub}}(k,n)$ be the reliability of the connected-(1,2)-or-(2,1)-out-of-(k,n):F lattice subsystem. Here k means the number of rows of the subsystem, namely the division unit, and mod (a,b) represents the remainder when a is divided by b. By using the above notations, the lower bound $LB(k)$ and the upper bound $UB(k)$ for the connected-(1,2)-or-(2,1)-out-of-(m,n):F lattice system are obtained as the following theorem:

Theorem

(a) $$LB(k) = R_{\text{sub}}(k,n)^{\left\lfloor \frac{m-k+1}{k-1} \right\rfloor + 1} \times R_{\text{sub}}(l,n), \tag{10}$$

where $l = \text{mod}(m-k+1, k-1)$ ($0 \le l \le k-1$) and when $l = 0$, $R_{\text{sub}}(0,n) = 1$.

(b) $$UB(k) = R_{\text{sub}}(k,n)^{\left\lfloor \frac{m}{k} \right\rfloor} \times R_{\text{sub}}(l,n), \tag{11}$$

where $l = \text{mod}(m,k)$ ($0 \le l \le k-1$) and when $l = 0$, $R_{\text{sub}}(0,n) = 1$. Theorems (a) and (b) are derived from ideas of Yamamoto and Miyakawa (1995) or Malinowski and Preuss (1996). The proof of Theorems are omitted.

As parameter k increases, more calculating time is required, and, on the other hand, the errors between the exact system reliability and bounds are smaller. Hence, taking into account the subsystem in Theorem (a) and (b), we should try to balance the quality of the bounds and the computational complexity of them. Namely, this is the trade-off problem.

4. NUMERICAL EXAMPLES

In this section, we perform numerical examples in order to compare with the existing bounds proposed by Yamamoto (1996).

In Table 3, we present, for a various of choices of m, n, p, our lower bounds $LB(k)$ (for $k=8,12,16$) [calculated by using Eq. (10)] and our upper bounds $UB(k)$ (for $k=8,12,16$) [calculated by using Eq. (11)]. Column labeled as Exact contains the exact value of the reliability of the connected-(1,2)-or-(2,1)-out-of-(m,n):F lattice system, given by Eq. (5). We compare these bounds with the lower bound derived by Yamamoto (1996), labeled LB_Y and the upper bound derived by Yamamoto (1996), labeled UB_Y. As previously described, if we select the event B_τ in Eq. (9) well, we can obtain the better upper bound. By using the optimal B_τ, the upper bound for the connected-(1,2)-or-(2,1)-out-of-(m,n):F lattice system is given as follow:

Table 3: Comparison of the lower and upper bounds.

m	n	p	LB_Y	$LB(8)$	$LB(12)$	$LB(16)$	Exact	UB_Y	$UB(8)$	$UB(12)$	$UB(16)$
10	10	0.90	0.16381	0.21231	—	—	0.23185	0.30014	0.24724	—	—
10	10	0.95	0.63727	0.65531	—	—	0.67016	0.69148	0.68338	—	—
10	10	0.99	0.98216	0.98173	—	—	0.98262	0.98284	0.98355	—	—
15	15	0.90	0.01468	0.02956	0.02956	—	0.03387	0.06400	0.03728	0.03728	—
15	15	0.95	0.34948	0.38115	0.38115	—	0.39462	0.42646	0.40631	0.40631	—
15	15	0.99	0.95887	0.95860	0.95860	—	0.95995	0.96050	0.96131	0.96131	—
30	30	0.95	0.01284	0.01849	0.01870	0.02009	—	0.03033	0.02569	0.02424	0.02287
30	30	0.98	0.49851	0.49394	0.50552	0.51140	—	0.52825	0.53431	0.52860	0.52295
30	30	0.99	0.84029	0.83945	0.83948	0.84192	—	0.84651	0.85157	0.84916	0.84676
50	50	0.98	0.17233	0.13920	0.14476	0.14762	—	0.16621	0.17427	0.16815	0.16517
50	50	0.99	0.61261	0.60313	0.60907	0.61206	—	0.62569	0.63897	0.63296	0.62998

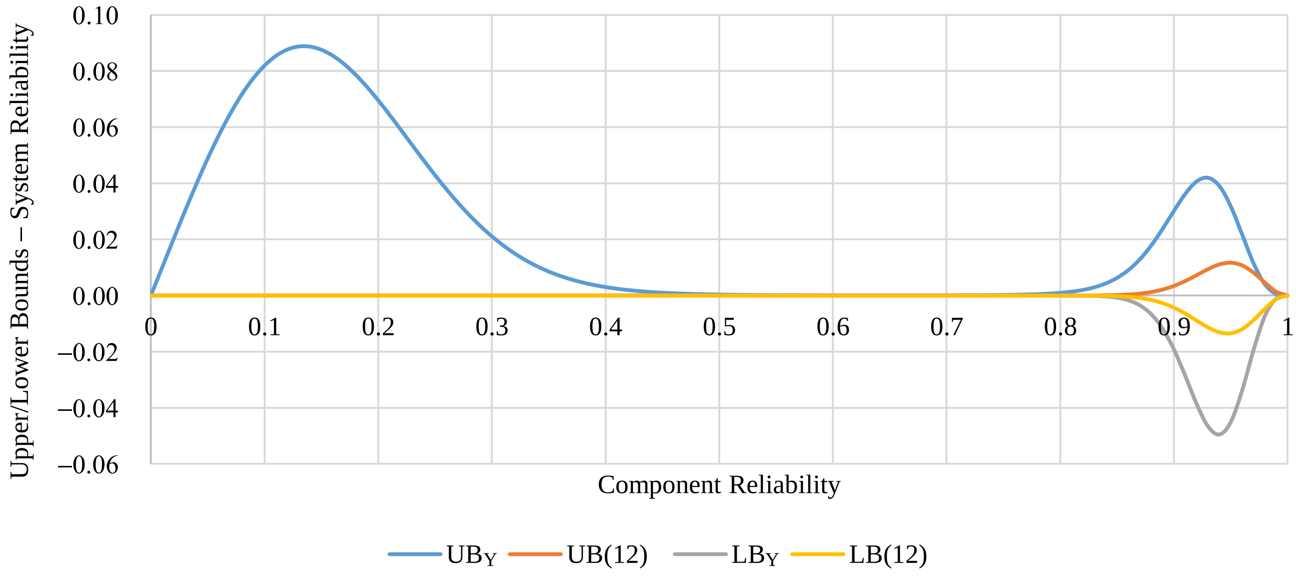

Figure 2: The difference between the exact system reliability and lower/upper bounds in the connected-(1,2)-or-(2,1)-out-of-(15,15):F lattice system

$$UB_Y = (1-2q^2+q^3)\left[1-p^4(2q^2-q^3)\right]^{(m-2)(n-2)} \cdot \left[1-p^2(2q^2-q^3)\right]^{m+n-4} \cdot (1-p^2q^2)^2(1-p^3q^2)^{m+n-4}. \quad (12)$$

For each system, it took about less than one second to calculate each of LB_Y, $LB(8)$, $LB(12)$, UB_Y, $UB(8)$ and $UB(12)$ with a Windows 10 Intel Core i5 3.20 GHz 4 GB, and MATLAB R2016a. On the other hand, $LB(16)$ and $UB(16)$ require approximately ten seconds as calculating time.

Table 3 shows that there are no best upper and lower bounds for all system. The proposed bounds tend to be better when the system size is small and component reliability p is not nearly equal to one. This tendency is common between proposed upper and lower bounds. Next, we investigate the relationship between the component reliability and the difference between the exact system reliability and upper/lower bounds. Fig. 2 presents Upper/Lower Bound − Exact Reliability as component reliability p for the connected-(1,2)-or-(2,1)-out-of-(15,15):F lattice system. From Fig. 2, the vicinity of $p = 0.95$, proposed bounds ($UB(12)$ and $LB(12)$) are better. However, if the component reliability p is very high (close to one), then the existing bounds (UB_Y and LB_Y) are better. Furthermore, Fig. 2 shows the vicinity of $p = 0.15$, UB_Y provides a bad value.

5. CONCLUSION

In this paper, we proposed new upper and lower bounds for evaluating the reliability of a connected-(1,2)-or-(2,1)-out-of-(m,n):F lattice system for the i.i.d. case. Then, we focus on the size of subsystem calculated easily. First, we clarify the system size which we can calculate within a set time by using Markov-based methods proposed by Nakamura *et al.* (2016). Next, we proposed new bounds based on the subsystem calculated by the Markov-based methods and perform numerical examples in order to compare with the existing bounds proposed by Yamamoto (1996). The result of the experiment shows that there are no best bounds for any system, and in the case that the system size is small and component reliability p is not nearly equal to one, the proposed bounds tend to be better. However, as parameter k increases, the errors between the exact system reliability and bounds are smaller, though more calculating time is required. Hence, the proposed methods give better bounds by making maximal use of available calculating time. Most pervious bounds are compared based on the error. Taking account into the error and calculating time, we will derive improved bounds for a connected-(1,2)-or-(2,1)-out-of-(m,n):F lattice system in the future.

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