Optimal inventory policy for items with power demand, backlogging and discrete scheduling period

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Abstract. In this paper an inventory system for items with a power demand pattern and backlogged shortages is developed. A known fixed time period is assumed and, moreover, the inventory cycle must be a multiple of that period. The holding cost, the backlogging cost and the ordering cost are the relevant costs considered in the inventory system management. The formulation of the inventory problem leads to a non-linear integer mathematical programming problem. An algorithmic procedure to calculate the economic order quantity and the optimal scheduling period that minimize the total cost per inventory cycle is proposed. Some numerical examples are solved to illustrate the theoretical results.

Keywords: EOQ inventory models, Backlogged shortages, Power demand pattern, Discrete scheduling period.

1. INTRODUCTION

The basic inventory models suppose that demand rate is constant. However, generally demand for items varies with time and inventory models must consider that situation. Thus, it is more realistic to analyze inventory systems where demand changes with time. It would allow to model appropriately the behavior and evolution of the inventory of products. In the literature of inventory theory, models involving time variable demand have received attention from several researchers. Thus, Silver and Meal (1973) established an approximate approach for a deterministic inventory system with time-dependent demand. Donaldson (1977) presented an algorithm for solving the classical no-shortage inventory policy for a linear trend in demand over a fixed time period. Ritchie (1984) analyzed the economic order quantity (EOQ) model with linear increasing demand. Dave (1981) studied a lot-size inventory model with a linear trend in demand and allowing shortages. Bahari-Kashani (1989) studied the optimal replenishment for deteriorating items with time-proportional demand. Goswami and Chaudhuri (1991) developed an inventory model, assuming shortages and linear demand. Hariga and Goyal (1995) analyzed an inventory model with time-varying demand rate, shortages and considering effects of inflation. Bose et al. (1995) analyzed an EOQ model for deteriorating items with a linear positive trend in demand and shortages backlogged. Bhunia and Maiti (1998) studied an inventory system for deteriorating items considering replenishment cost dependent on lot-size and linear trend in demand. Chang and Dye (1999) discussed an EOQ model for deteriorating items with time varying demand and partial backlogging. Wu (2002) discussed also an EOQ model with time-varying demand, considering deterioration and shortages. Yang et al. (2004) presented an approach to analyze an inventory system with non-linear decreasing demand. Sakaguchi (2009) developed an inventory policy for a system with time-varying demand. Omar and Yeo (2009) studied an inventory system that satisfied a continuous time-varying demand. Mishra and Singh (2010) developed an inventory model with constant rate of deterioration and time dependent demand.

The demand patterns are referred to as different ways by which products are taken out of inventory to supply customer demand. If the demand rate is the same during all the inventory cycle, the demand pattern is known as uniform demand pattern. However, there are other ways by which the units may be withdrawn throughout the period. The power demand pattern allows to suit demand to more practical situations. Thus, this pattern can not only represent the behavior of demand when is uniformly distributed throughout the period, but also models situations where a high percentage of units may be withdrawn at the beginning or at the end of the period.

Several papers on inventory systems consider that the demand follows a power pattern. Thus, Goel and Aggarwal (1981) studied an order-level inventory system with power demand pattern for deteriorating items. Datta and Pal (1988) analyzed an inventory model with power demand pattern and variable rate of deterioration. Lee and Wu (2002) presented an inventory model for items with deterioration, shortages and power demand pattern. Dye (2004) extended this last model to a general class with time-proportional backlogging rate. Singh et al. (2009) analyzed an EOQ model for perishable items with power demand pattern and partial backlogging. Rajeswari and Vanjikkodi (2011) studied an inventory model for deteriorating items with partial backlogging and power demand pattern. Recently, Mishra and Singh (2013) presented an economic order quantity model for deteriorating items with power demand pattern and shortages partially backlogged.

In all the above papers the scheduling period is a fixed period. Thus, the inventory total cost depends on a single decision variable. If the inventory system is without shortages, the decision variable is either the lot size or the initial stock level. On the other hand, when the inventory system allows shortages, the decision variable is time where the net stock level is zero.

However, in the model here presented the inventory total cost will depend on two decision variables. Thus, we develop an inventory model for items with power demand on a known basic period. The inventory cycle must be a multiple of that period and the lot size must be also a multiple of the total quantity demanded along that basic period. Shortages are allowed and completely backlogged. The holding cost, the backlogging cost and the ordering cost are the three costs considered in the system management. The objective of the inventory problem consists of determining the optimal scheduling period and the economic lot size such that the total inventory cost per unit time is minimized. The formulation of this objective leads to a non-linear integer mathematical programming problem. An algorithmic procedure to calculate the economic order quantity and the optimal scheduling period that minimize the total cost per inventory cycle is proposed.

The organization of the paper is as follows. In Section 2 we introduce the notation used throughout of the paper and the basic assumptions considered for the inventory system. In Section 3, the mathematical model that describes the inventory problem is formulated. An algorithm to solve the inventory problem and determine the optimal policy is presented in Section 4. Some numerical examples are provided in Section 5 to illustrate the solution procedure. Finally, conclusions and future research are commented.

2. HYPOTHESIS AND NOTATION

We will use throughout the paper the following notation.

τ	Basic time period. That period is fixed and known.
D	Demanded total quantity along the basic period τ. That quantity is known.
r	Average demand per basic period ($r = D/\tau$).
f(t)	Demand up to time t ($0 \le t \le \tau$) along each basic period. Note that $f(\tau) = D$.
n	Demand pattern index (n>0).
T	Length of inventory cycle or scheduling period. That period T must be a multiple of the basic period τ.
k	Number of basic periods that contains the inventory cycle ($T = k\tau$).
S	Initial stock level or order level. That inventory level S must be a multiple of the demand D.
m	Integer positive such that S = mD.
s	Reorder point. The inventory should be replenished when s = (m-k)D.
Q	Lot size or order quantity. The lot size must be equal to Q = kD.
I(t)	Net inventory (on hand - backorders) level at time t ($0 \le t \le T$).
t_0	Time at which the inventory level reaches zero.

A	Ordering cost.
H	Holding cost per unit and per time unit.
W	Backlogging cost per unit and per time unit.
$I_1(k,m)$	Average amount carried in inventory.
$I_2(k,m)$	Average shortage in inventory.
$I_3(k)$	Number of replenishments per unit of time.
$C_1(k,m)$	Holding cost per unit of time.
$C_2(k,m)$	Backlogging cost per unit of time.
$C_3(k)$	Ordering cost per unit of time.
$C(k,m)$	Total cost per unit of time of the inventory system.

The inventory system is based on the following assumptions:

A single item is considered in the inventory system.

The planning horizon is infinite.

The fluctuations of the net stock level are repeated on time along each inventory cycle.

The replenishment rate is infinite.

Lead time is zero or negligible.

Shortages are allowed and completely backlogged.

Demand during the basic period τ follows a power demand pattern. The average demand r (i.e., r = D/τ) per basic period is deterministic, but the manner in which quantities are extracted of inventory depends on the time when they are removed. This way by which demand occurs during the period τ is known as the demand pattern. Thus the demand f(t) up to time t ($0 \le t \le \tau$) varies with time and is assumed to be $f(t) = rt^{1/n}/\tau^{(1-n)/n}$, where r is the average demand and n is the pattern index, with $0<n<\infty$. The demand rate at time t ($0 \le t \le \tau$) is $f'(t) = rt^{(1-n)/n}/n\tau^{(1-n)/n}$.

This pattern in the demand is called the power demand pattern (see Naddor, 1966; Datta and Pal, 1988; Lee and Wu, 2002; Rajeswari and Vanjikkodi, 2011; Sicilia et al., 2012; and Sicilia et al., 2013).

The decision variables are the integer values k and m, with $1 \le m \le k$. If we determine those values, then we obtain the initial stock level S = mD, the scheduling period T = kτ, the reorder point s = (m-k)D and the lot size Q = kD. The input parameters of the inventory system are D, τ, n, h, w and A.

The aim consists of determining the optimal scheduling period, the economic ordering quantity (EOQ) and the optimal reorder point that optimize the management of the inventory system described by the previous hypotheses. In the following section we present a mathematical model to study the inventory policy.

3. MATHEMATICAL FORMULATION OF THE INVENTORY PROBLEM

Let us consider an inventory system in which there is a basic period τ during which the behavior of customers' demand follows a power pattern with index n. An example of such a situation occurs for products as coffee, milk, breads, croissants, fruit, newspapers, journals, etc. In these cases the demands of these goods toward the beginning of the day (basic period) are greater that on the afternoon or evening. Those demands are modeled by using a power demand pattern with index n > 1. Another situation occurs when the demand of a product toward the beginning of the day is smaller than the demand at the end of the day. That situation occurs, for example, for items as pizzas, hamburgers, hot dogs, cinema tickets, popcorns, etc. In these cases demands are modeled by a power demand pattern with index n < 1.

The replenishment of the inventory is made every T time units. That inventory cycle or scheduling period T must be a multiple of the basic period τ. Hence, T = kτ, being k a positive natural number. As the known demand over the basic period is D, then along every inventory cycle the total demand will be kD units. Therefore, the replenishment size or lot size must be Q = kD units, because the ordering quantity to replenishment the inventory must be equal to the total demand along the inventory cycle.

Let I(t) be the net stock (on hand - backorders) level at time t ($0 \le t \le T$). At the beginning of the inventory cycle, the initial net stock level is S units. That level must be a multiple of the D units because that quantity represents the demand along the basic period τ. Hence, we have that S = mD, being m a integer number. Thus, at time t_0 = mτ, the inventory level attains a level zero. During the scheduling period [0,T] the stock level declines due to demand of items. At t = T, the inventory attains a level s (reorder point). That reorder point is determined by the difference S – Q. Hence, the reorder point is a value given by s = (m-k)D units. Next, the inventory is filled with a replenishment quantity or lot size Q = kD units, which raises the inventory stock up to level S and a new inventory cycle starts again. Note that in each inventory cycle the lot size Q is equal to the total demand rT because k = T/τ and r = D/τ.

The inventory cycle, or time period, T can be partitioned in k time intervals of equal length τ. Thus, we have

$$[0,T]=[0,\tau]\cup[\tau,2\tau]\cup....\cup[(k-1)\tau,k\tau=T]$$

Under the assumptions previously commented in Section 2, the differential equation that describe the fluctuations of the net stock level I(t) over every time interval ((i-1)τ, iτ), with i = 1,2,...,k, is

$$\frac{dI(t)}{dt}=-\frac{r[t-(i-1)\tau]^{(1-n)/n}}{n\tau^{(1-n)/n}} \qquad (1)$$

The boundary conditions are I(0) = S and I(iτ)=(m-i)D, for i=1,2,…,k-1. Note that these conditions imply that $I(t_0)$ = 0 and I(T) = s.

The solution of the above differential equation over the time interval [0,τ] leads to

$$I_1(t) = S - \frac{r}{\tau^{(1-n)/n}} t^{1/n}, \quad \text{if} \quad 0 \le t \le \tau \tag{2}$$

For i = 2,...,k, the equation of the inventory level over the time period [(i-1)τ, iτ] is

$$I_i(t) = I((i-1)\tau) - \frac{r}{\tau^{(1-n)/n}} [t-(i-1)\tau]^{1/n}, \quad \text{if} \quad (i-1)\tau \le t < i\tau \tag{3}$$

Thus, the inventory level function I(t) over the scheduling period [0,T=kτ] is given by

$$I(t) = I_i(t) = (m-i+1)D - \frac{r}{\tau^{(1-n)/n}} [t-(i-1)\tau]^{1/n}, \quad \text{if} \quad (i-1)\tau \le t < i\tau, \quad \text{for} \quad i = 1,2,...,k. \tag{4}$$

Note that the net stock level I(t) is always a continuous and decreasing function on the time period [0,T). In addition, the decreasing evolution of the stock level is the same over every time interval [(i-1)τ, iτ], for i = 1,2,…,k.

Both the average amount carried in inventory $I_1(k,m)$ and the average shortage $I_2(k,m)$ depend on the positive integer variables k and m.

As 1 ≤ m ≤ k, then some inventories are carried and also shortages can occur. At time $t = t_0 = m\tau$ the inventory stock attains a level zero. Thus, along the interval $[t_0,T)$, shortages occur in the inventory system and they are backlogged at the end of period.

The total amount carried in inventory during the period $[0,t_0]$ is the sum of the quantities carried in inventory along of periods [(i-1)τ, iτ], for i = 1,2,…,m. That total amount is

$$\sum_{i=1}^{m} \int_{(i-1)\tau}^{i\tau} I_i(t)dt = mD\tau\left(\frac{m(n+1)+1-n}{2(n+1)}\right) \tag{5}$$

Thus, the average amount carried in inventory during the period [0,T] is given by the expression

$$I_1(k,m) = \frac{1}{T}(mD\tau)\left(\frac{m(n+1)+1-n}{2(n+1)}\right) = \frac{mD}{k}\left(\frac{m(n+1)+1-n}{2(n+1)}\right) \tag{6}$$

Now, we have to calculate the shortages during the inventory cycle. It is determined by the sum of the shortages obtained in every time interval [(i-1)τ,iτ], for i =m+1,..., k. Thus, we have

$$\sum_{i=m+1}^{k} \int_{(i-1)\tau}^{i\tau} [-I_i(t)]dt = (k-m)D\tau\left(\frac{(k-m)(n+1)+n-1}{2(n+1)}\right) \tag{7}$$

Hence, the average shortage in the inventory during the period [0,T] is

$$I_2(k,m) = \frac{1}{T}(k-m)D\tau\left(\frac{(k-m)(n+1)+n-1}{2(n+1)}\right) = \frac{(k-m)D}{k}\left(\frac{(k-m)(n+1)+n-1}{2(n+1)}\right) \tag{8}$$

Finally, the average number of replenishments per unit of time is always $I_3(k) = 1/T = 1/(k\tau)$.

In the following paragraphs, we determine the costs subject to control in the inventory system. The inventory total cost per unit time consists of the following cost components: holding cost, backlogging cost, and ordering cost.

The holding cost per unit of time is

$$C_1(k,m) = h\frac{mD}{k}\left(\frac{m(n+1)+1-n}{2(n+1)}\right) \tag{9}$$

The backlogging cost per unit of time is

$$C_2(k,m) = w\frac{(k-m)D}{k}\left(\frac{(k-m)(n+1)+n-1}{2(n+1)}\right) \tag{10}$$

The ordering cost per unit of time is

$$C_3(k) = \frac{A}{T} = \frac{A}{k\tau} \tag{11}$$

The total inventory cost per unit of time is the sum of the above costs. Hence, that total cost is given by

$$C(k,m) = (h+w)\frac{mD}{k}\left(\frac{m(n+1)+1-n}{2(n+1)}\right) + wD\left(\frac{(k-2m)(n+1)+n-1}{2(n+1)}\right) + \frac{A}{k\tau} \tag{12}$$

Thus, we have to solve the following non-linear integer mathematical programming problem:

$$\text{Min} \quad C(k,m) = (h+w)\frac{mD}{k}\left(\frac{m(n+1)+1-n}{2(n+1)}\right) + wD\left(\frac{(k-2m)(n+1)+n-1}{2(n+1)}\right) + \frac{A}{k\tau} \tag{13}$$

$$\text{subject to} \quad k \text{ and } m \text{ are integer variables with } 1 \le m \le k$$

4. SOLVING THE INVENTORY PROBLEM

To find the solution of the inventory problem, we have to seek the minimum of the function C(k,m) for the range 1≤m≤k, with k and m integer variables.

Relaxing the integer restrictions on the variables, and assuming that now k and m are continuous variables, then we could use the differential calculus to obtain the optimal solution for the continuous inventory problem.

Differentiating with respect to m, we have

$$\frac{\partial C}{\partial m}=\frac{(h+w)mD}{k}+\frac{(h+w)D(1-n)}{2k(n+1)}-wD \tag{14}$$

and differentiating with respect to k, we obtain

$$\frac{\partial C}{\partial k}=-\frac{(h+w)mD}{k^2}\left[\frac{m(n+1)+1-n}{2(n+1)}\right]+\frac{wD}{2}-\frac{A}{k^2\tau} \tag{15}$$

To find the minimum point (k_0,m_0), we have to solve the equations

$$\frac{(h+w)mD}{k}+\frac{(h+w)D(1-n)}{2k(n+1)}-wD\ =0 \tag{16}$$

$$-\frac{(h+w)mD}{k^2}\left[\frac{m(n+1)+1-n}{2(n+1)}\right]+\frac{wD}{2}-\frac{A}{k^2\tau}\ =0 \tag{17}$$

From (16), we obtain m as a function of k, that is,

$$m=m(k)=\frac{kw}{h+w}-\frac{(1-n)}{2(\mathrm{n}+1)} \tag{18}$$

Substituting m = m(k) in equation (17), and solving that equation with respect to variable k, we have

$$k_0=\sqrt{\frac{2A(h+w)}{hw\tau D}-\frac{(h+w)^2(1-n)^2}{4hw(n+1)^2}} \tag{19}$$

From (18), we obtain

$$m_0=\sqrt{\frac{2Aw}{h\tau(h+w)D}-\frac{w(1-n)^2}{4h(n+1)^2}}-\frac{1-n}{2(n+1)} \tag{20}$$

Substituting k_0 and m_0 given in (19) and (20) in the cost function (12), we have the cost

$$C_0=C(k_0,m_0)=\sqrt{\frac{hwD}{h+w}\left[\frac{2A}{\tau}-\frac{(h+w)D(1-n)^2}{4(n+1)^2}\right]} \tag{21}$$

Note that the terms inside the roots in (19), (20) and (21) should be positive values. Thus, the condition

$$8A(n+1)^2>(h+w)D\tau(1-n)^2 \tag{22}$$

must be satisfied.

In addition, an easy computation shows that the Hessian at point (k_0,m_0) is positive when (22) is true.

Therefore, if the condition (22) is satisfied, then (k_0,m_0) is a minimum point of the function C(k,m) given in (12) when the variables k and m are continuous variables.

If the variable m is fixed, the following result shows that the cost function C(k,m) has a minimum point on the straight line determined by m.

Theorem 1. *Consider m and k as continuous variables on the region Ω = {(k,m)/ k>0, m≥1}. Fixed m, with m≥1, the cost function $C_m(k)$ = C(k,m) is a convex function with respect to the variable k and has a minimum point at the value k_m given by*

$$k_m\ =\sqrt{\frac{(h+w)mD\tau[(m-1)n+m+1]+2A(n+1)}{wD\tau(n+1)}} \tag{23}$$

Proof. The proof is immediate. □

If we substitute k_m in the function C(k,m) given in (12), then we obtain a cost function that depends of m only, that is

$$C(m)=C(k_m,m)=(h+w)\frac{mD}{k_m}\left(\frac{m(n+1)+1-n}{2(n+1)}\right)+wD\left(\frac{(k_m-2m)(n+1)+n-1}{2(n+1)}\right)+\frac{A}{k_m\tau} \tag{24}$$

In the following result we study how is the behavior of the function C(m) given by (24).

Theorem 2. *Let us assume that m is a continuous variable.*

(a) If the condition (22) is true, then C(m) = C(k_m,m) is a convex function on the region characterized by m≥1.

(b) If (22) is false, then C(m) = C(k_m,m) is a concave and increasing function on the region determined by m≥1.

Proof. Substituting (23) into (24) and differentiating C(m) with respect to m, we have

$$\frac{\partial C(m)}{\partial m}=\frac{2m(n+1)+1-n}{2(n+1)}\sqrt{\frac{(h+w)^2wD^3\tau(n+1)}{(h+w)mD\tau[(m-1)n+m+1]+2A(n+1)}}\ -wD \tag{25}$$

Note that the above derivative is positive if, and only if, the equation $m^2+bm+c>0$, being

$$b=\frac{1-n}{1+n}\quad\text{and}\quad c=\frac{(n-1)^2(h+w)}{4(n+1)^2h}-\frac{2Aw}{Dh\tau(h+w)}. \tag{26}$$

As the discriminate of the above equation is

$$\Delta=b^2-4c=\frac{[8A(n+1)^2-(h+w)(n-1)^2D\tau]w}{Dh\tau(h+w)(n+1)^2}$$

We have the following cases:

- If $8A(n+1)^2=(h+w)D\tau(1-n)^2$, then $\Delta=0$. Thus, for all m ≥ 1, we have

  $$m^2+bm+c=\left(m-\tfrac{n-1}{2(n+1)}\right)^2>0$$

  Hence, in this case, the derivative $\partial C(m)/\partial m>0$ and C(m) is an increasing function on the interval [1,∞).
- If $8A(n+1)^2<(h+w)D\tau(1-n)^2$, then $\Delta<0$ and $m^2+bm+c>0$ always. Therefore, in this case, $\partial C(m)/\partial m>0$ and C(m) is increasing for all m.
- If $8A(n+1)^2>(h+w)D\tau(1-n)^2$, then $\Delta>0$ and the equation $m^2+bm+c=0$ has two real roots. Thus, in this case, the function C(m) is decreasing for $m\le m_0$ and is

increasing for $m \geq m_0$, being m_0 the value given by (20).

In addition, the second derivative of C(m) is

$$\frac{\partial^2 C(m)}{\partial m^2} = \frac{D^2 \tau w(h+w)[8A(n+1)^2 - (h+w)(n-1)^2 D\tau]}{4[(h+w)mD\tau[m(n+1)+1-n]+2A(n+1)]^2} k_m \quad (27)$$

As $k_m > 0$ for all $m \geq 1$, then the sign of the second derivative depends on whether the condition (22) is or not is true. Thus, if $8A(n+1)^2 > (h+w)D\tau(1-n)^2$, then that second derivative is positive and the function C(m) is convex for all m.

Otherwise, the function C(m) is increasing and concave for $m \geq 1$.

Corollary 1. *Let us consider that m is a continuous variable.*

(a) If the condition (22) is true, then the minimum point of the function $C(m) = C(k_m,m)$ on the region Ω characterized by $m \geq 1$ is given by the formula (20).

(b) If (22) is false, then the minimum point of the function $C(m) = C(k_m,m)$ on the region Ω is m=1.

Proof. It follows easily from Theorem 2. □

Taking into account the above results, we can develop a procedure to obtain the optimal integer solution for the problem shown in (13), which is exposed in the following section.

4.1 Optimal approach

A simple procedure to give an optimal integer solution for the inventory problem is below presented.

4.1.1 Algorithm

Step 1 Set m = 1. Using the formula (23), calculate the point

$$k_1 = \sqrt{\frac{2(h+w)D\tau + 2A(n+1)}{wD\tau(n+1)}} \quad (28)$$

- If $k_1 \leq 1$, then choose as initial solution $(k^*,m^*) = (1,1)$. Compute $C(k^*,m^*) = C(1,1)$ by using the formula (12). Go to step 4.
- Otherwise, go to step 2.

Step 2 Determine the largest integer not greater than k_1 and the smallest integer not less than k_1, that is, the integers $k_{11} = \lfloor k_1 \rfloor$ and $k_{12} = \lceil k_1 \rceil$. Compute the costs $C(k_{11},1)$ and $C(k_{12},1)$.
- If $C(k_{11},1) \leq C(k_{12},1)$, then choose the value $k'_1 = k_{11}$. Go to step 3.
- Otherwise, consider $k'_1 = k_{12}$. Go to step 3.

Step 3 Set as initial solution $(k^*,m^*) = (k'_1,1)$ with cost $C(k^*,m^*) = C(k'_1,1)$. Go to step 4.

Step 4 Set m = 2. Calculate the point k_2 by using the formula (23), that is

$$k_2 = \sqrt{\frac{2(h+w)D\tau(n+3) + 2A(n+1)}{wD\tau(n+1)}} \quad (29)$$

Compute the cost $C(k_2,m=2)$ by using the formula (24). Go to step 5.

Step 5 If $8A(n+1)^2 > (h+w)D\tau(1-n)^2$, go to step 6. Otherwise, go to step 9.

Step 6 Obtain the optimal solution (k_0,m_0) given by (19) and (20) for the inventory problem with continuous variables. Calculate the minimum cost C_0 given by (21). Go to step 7.

Step 7 If $m < m_0$, then go to step 10. Otherwise, go to step 9.

Step 8 Put m = m+1. Calculate the point k_m by using the formula (23). Determine the cost $C(k_m,m)$ by using the formula (24). Go to step 9.

Step 9 If $C(k^*,m^*) \leq C(k_m,m)$, then the optimal solution is the point (k^*,m^*) and the minimal integer cost is $C(k^*,m^*)$. Stop. Otherwise, go to step 10.

Step 10 Determine the largest integer not greater than k_m and the smallest integer not less than k_m, that is, the integers $k_{m1} = \lfloor k_m \rfloor$ and $k_{m2} = \lceil k_m \rceil$. Compute the costs $C(k_{m1},m)$ and $C(k_{m2},m)$.
- If $C(k_{m1},m) \leq C(k_{m2},m)$, then choose the value $k'_m = k_{m1}$. Save the point (k'_m,m) and the cost $C(k'_m,m)$. Go to step 11.
- Otherwise, choose $k'_m = k_{m2}$. Keep the point (k'_m,m) and the cost $C(k'_m,m)$. Go to step 11.

Step 11 Compare the costs of the points (k^*,m^*) and (k'_m,m).
- If $C(k^*,m^*) \leq C(k'_m,m)$, then keep the point (k^*,m^*) and remove the point (k'_m,m). Go to Step 8.
- Otherwise, save the point (k'_m,m). Set $(k^*,m^*) = (k'_m,m)$ and $C(k^*,m^*) = C(k'_m,m)$. Back to Step 8.

It is interesting to compare the solution (k^*,m^*) obtained by the above algorithm with the solution (k_0,m_0) in which no integer constraint is imposed on the m and k values. Note that $C^* \geq C_0$ always, and the gap between C^* and C_0 gives a measure of the goodness of the integer solution. Thus, the ratio $\gamma = (C^* - C_0)/C_0$ can be used as a measure of the approximation between the integer and continuous solutions.

5. NUMERICAL EXAMPLES

In this section we present several numerical examples to illustrate the theoretical results.

Example 1. Let us consider the following parametric values for an inventory system with appropriate units: the basic period is τ = 1 week, demand is D = 40 units, average demand is r = 40 units per week, A = \$ 600 per order, h = \$ 4 per unit per week, w = \$ 2 per unit per week, and the demand pattern index n = 1/2. Firstly, from (28), we calculate the point k_1= 4.35890. Then, we consider the integers $k_{11} = \lfloor k_1 \rfloor = 4$ and $k_{12} = \lceil k_1 \rceil = 5$. Now, we compute the costs C(4,1) = \$ 256.667 and C(5,1) = \$ 258.667. As C(4,1) < C(5,1), the initial solution is (k*,m*) = (4,1). Next, from (29), we calculate k_2 = 5.38516 and compute the cost $C(k_2,2)$ = \$ 257.480. Then, we check that the condition (22) is satisfied. Following the algorithm given in Section 4, we calculate the values m_0 and k_0 given in (19) and (20). Thus, we obtain k_0 = 4.73022 and m_0 = 1.41007. As m = 2 is greater than m_0, in Step 9, we compare C(k*,m*) with $C(k_2,2)$. Thus, the optimal solution is C(k*,m*) = (4,1) with cost C* = C(4,1) = \$ 256.667. Hence, the scheduling period is T* = 4 weeks and the lot size is Q* = 160 units. From (21), the minimum cost for the continuous inventory problem is C_0 = \$ 252.279. Hence, the measure of the goodness of the optimal integer solution with respect to the continuous one is γ = 0.0173941. It means that the integer solution is very close to the minimum cost of the continuous optimal policy.

Example 2. We consider now the same input parameters than the above example, but only changing the index of the power demand pattern to n = 3. From (28), we have k_1 = 4.06202. Next, we calculate C(4,1) = \$ 265; and C(5,1) = \$ 272. As the lower cost corresponds to (4,1), the initial solution is (k*,m*) = (4,1). Then, from (29), we obtain k_2 = 4.89898 with cost $C(k_2,2)$ = \$ 251.918. The condition (22) is again satisfied. Thus, from (19) and (20), we have k_0 = 4.71368 and m_0 = 1.82123. In this case, the minimum cost of the continuous inventory problem is C_0 = \$ 251.396. As m = 2 > m_0, we compare C(k*,m*) = \$ 265 with $C(k_2,2)$ = \$ 251.918. As C(k*,m*) > $C(k_2,2)$, then we calculate the costs C(4,2) = \$ 260 and C(5,2) = \$ 252. Keep the point (5,2) with cost C(5,2) = \$ 252. Now, the new solution is (k*,m*) = (5,2). Next, we set m = 3, calculate k_3 = 6.12372 and the cost $C(k_3,3)$ = 269.898. As C(5,2) = \$ 252 < $C(k_3,3)$, then the optimal integer solution is the point (5,2) with cost C(5,2) = \$ 252. The inventory cycle is T* = 5 weeks and the lot size is Q* = 200 units. As can be seen, the cost of the optimal integer policy is very close to the continuous one.

Example 3. Consider the same input parameters than the Example 1, but only changing demand to D = 8000 units. We start calculating the point k_1 = 2.01866. Next, we calculate the costs C(2,1) = \$ 13633.33 and C(3,1) = \$ 16200. As the lower cost corresponds to (2,1), that point is the initial solution, that is, (k*,m*) = (2,1). We set m = 2 and calculate k_2 = 3.75167 with cost $C(k_2,2)$ = \$ 25359.99. In this case, the condition (22) is false. As C(2,1) < $C(k_2,2)$, then the optimal solution is the point (k*,m*) = (2,1) and the minimal integer cost is C(2,1) = \$ 13633.33. Hence, the inventory cycle is T* = 2 weeks and the lot size is Q* = 16000 units.

6. CONCLUSIONS AND FUTURE RESEARCH

In this paper we develop an inventory model for a single product where the demand over a basic time period follows a power demand pattern. The shortages are allowed and will be filled with the arrival of a new lot of products.

The replenishment of the inventory is made every scheduling period, being that period a multiple of the basic period. Also, the lot size or ordering quantity to replenishment the inventory must be equal to a multiple of the demand along the basic period.

We have proposed an approximate approach to calculate the inventory policy. Thus, we have presented an algorithmic procedure to determine easily the inventory policy, that is, the scheduling period, the economic lot size and the inventory cost.

A future study will incorporate in the proposed model new assumptions such as deterioration of the products, replenishment non-instantaneous, and/or lost sales cost.

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