On the Convex Relaxation for Linear Complementarity Constraints

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Abstract. We discuss the problem of generating strong cutting planes for linear programs with linear complementarity constraints (LPCCs). In particular, we exploit complementarity constraints to derive cuts from the optimal simplex tableaux of LP relaxations of the problem. We introduce cmax procedure to derive convex hull for the corner relaxation and compare the strength of the cuts obtained by disjunctive programming. We also introduce the notions of split cut and split closure for LPCC problems with bounded variables and compare these notions with their integer programming counterpart.

Keywords: complementarity constraint, integer programming, disjunctive programming

1. INTRODUCTION

Linear programs with linear complementarity constraints (LPCCs) have applications in optimal control, economics, and engineering. Several practical applications are described in (Ferris and Pang,1997) along with a detailed literature review. LPCCs can be formulated as:

$$\max\left\{ f_x + g_y + h_z \mid (x,y,z)\in S^{\perp}\right\} \qquad (1)$$

where $S^{\perp}=\left\{(x,y,z)\in R^m\times R^n\times R^n \mid Ax+By+Cz\ge d, y\perp z\right\}$, with $f\in R^m$, $g\in R^n$, $h\in R^n$, $A\in R^{p\times m}$, $B\in R^{p\times n}$, $C\in R^{p\times n}$, and $d\in R^p$. Given $S^{\perp}$, we introduce

$$S=\left\{(x,y,z)\in R^m\times R^n\times R^n \mid Ax+By+Cz\ge d\right\}$$

to be the linear programming relaxation of $S^{\perp}$, and define $C_j=\left\{(x,y,z)\in R^m\times R^n\times R^n \mid y_i z_i=0, \forall i=1,..,j\right\}$ to be the set of solutions in R^{m+2n} that satisfy the j first complementarity constraints. Both S and C_n are relaxations of $S^{\perp}$, and it is clear that $S^{\perp}=S\cap C_n$. If (x^*,y^*,z^*) is an extreme point of S that does not belong to C_n, there exists an index $i\in N$ for which $y_i^* z_i^*>0$. In any basic solution corresponding to vertex (x^*,y^*,z^*), variables y_i and z_i will both be basic (if we consider variables y and z to not have explicit upper bounds). Let B and N be the index sets of basic and nonbasic variables of the corresponding basic solution. The simplex tableau corresponding to this basis contains the rows

$$\begin{aligned} y_i + \sum_{j\in N} a_j x_j &= a_0 \\ z_i + \sum_{j\in N} b_j x_j &= b_0, \end{aligned} \qquad (2)$$

Where variables x are nonbasic. Because we assume that the complementarity constraint on y_i and z_i is not satisfied, we must have that $a_0\ne 0$ and $b_0\ne 0$.Various IP-based methods have been used to derive the convex hull of LPCCs. In particular, LPCCs have feasible regions that are facially disjunctive. As a result, disjunctive programming, see (Balas, 1998), is a natural approach to study these problems. Several family of tableaux cuts for LPCCs have been introduced in the literature. Ibaraki (1973) presents a family of C-cuts for the simplex tableau of the LP relaxation of a LPCC where the complementarity constraint on basic variables y_i and z_i is not satisfied. The C-cut is obtained geometrically as the plane that passes through n points that are the first intersection points of the n rays (a_j,b_j) for $j\in N$ with the plane $y=0$ or $z=0$.

The C-cut is similar to cuts introduced by Gomory (1960) for mixed integer programs (MIPs). Audet et al (2007) introduce "simple cuts" for a linear complementarity formulation of bi-level linear programs. These cuts are also described Audet et al (2007). Although the above cuts have been studied numerically, current literature does not address if they are strong for the one-complementarity corner relaxation (1). It also does not address whether the convex hull of this relaxation can easily be constructed. We mention however that Hu et al (2008) describes a variant of the reformulation linearization technique developed in Sherali and Adams (1990) for 0-1 MIPs, to construct convex relaxations of LPCCs. Sequential convex relaxation methods to derive the convex hull of $S^{\perp}$ are investigated in Balas et al (1993) and Judice et al (2006). These algorithms use the fact that every cutting plane that is generated induces a distinct face of some member of a finite family of polyhedra. In this chapter, we study the relaxation of LPCCs defined by (2), *i.e.*, a two-row relaxation of a simplex tableau with complementarity basic variables y_i and z_i . This relaxation is obtained by (i) relaxing all tableau rows not corresponding to basic variables y_i and z_i , (ii) relaxing all complementarity requirements on non-basic variables, and (iii) relaxing the negativity constraints on the basic variables. This relaxation is analogous to the corner relaxation of a traditional MIP with a single integer variable obtained from a simplex tableau where this variable has fractional value. For this reason, we refer to this set as a one-complementarity corner relaxation of $S^{\perp}$. This paper is organized as follow. In Section 2, we present in more detail cuts for LPCCs that have been derived from simplex tableaux rows of the form (2) using a single complementarity constraint between y_i and z_i. We derive a closed-form linear description for the convex hull of the one-complementarity corner relaxation of a LPCC. For problems with multiple complementarity constraints, we introduce in Section 3 the notion of split cut and split closure for LPCCs with bounded variables. We conclude remarks in Section 4.

2. VALID INEQUALITIES FOR ONE-COMPLEMENTARITY SET

We first review family of tableaux cuts that have been proposed in the literature, and show that they often can be strengthen upon. Classically, these cuts have been obtained by focusing on relaxations of $S^{\perp}$ with two rows. In the foregoing discussion, we write these two rows as

$$y = a_0 - \sum_{j\in N} a_j x_j$$
$$z = b_0 - \sum_{j\in N} b_j x_j \qquad (3)$$

In (3), we use x to represent non-basic variables, and we use y and z to denote basic variables that must satisfy the complementarity requirement that $yz=0$. In the literature, tableaux cuts are often derived without using lower bounds on variables y and z. For this reason, we also do not require variables y and z to have lower or upper bounds in our relaxation. We assume without loss of generality that $a_0>0$ and $b_0>0$ by introducing variable $\bar{y}=-y$ instead of y if $a_0<0$, and by introducing variable $\bar{z}=-z$ instead of z if $b_0<0$. We are therefore interested in the convex hull of

$$\bar{P}=\left\{(x;y,z)\in R_+^n\times R^2 \mid y=a_0-\sum_{j\in N}a_jx_j, z=b_0-\sum_{j\in N}b_jx_j, yz=0\right\}$$

Observe that $\bar{P}=\left(P_1\times\{0\}\times R\right)\cup\left(P_2\times R\times\{0\}\right)$

$$P_1=\left\{x\in R_+^n \mid \sum_{j\in N}a_jx_j=a_0\right\},\quad P_2=\left\{x\in R_+^n \mid \sum_{j\in N}b_jx_j=b_0\right\}.$$

Since variables y and z are expressed as a linear combination of other variables, every valid inequality for $\bar{P}$ can be written solely in terms of the non-basic variables x_j for $j\in N$. For this reason, we next study valid inequalities for $clconv(\bar{P})$ in the space of non-basic variables x. Define $P=P_1\cup P_2$, we have following result.

Lemma 1. $proj\left(clconv(\bar{P})\right)=clconv(P)$.

We now derive strong valid inequalities for $clconv(P)$. To streamline the discussion, we introduce the following notation:

$$N_{++}=\{j\in N \mid a_j>0, b_j>0\}, N_{--}=\{j\in N \mid a_j<0, b_j<0\}$$
$$N_0=\{j\in N \mid a_jb_j=0\}$$
$$I_+^1=\{j\in N \mid a_j>0\}, I_-^1=\{j\in N \mid a_j<0\}$$
$$I_0^1=\{j\in N \mid a_j=0\}$$
$$I_+^2=\{j\in N \mid b_j>0\}, I_-^2=\{j\in N \mid b_j<0\}$$
$$I_0^2=\{j\in N \mid b_j=0\}$$

If $a_j\le 0$ for all $j\in N$ then $P_1=\emptyset$. In this case, $clconv(P)=P_2$. Similarly, if $b_j\le 0$ for all $j\in N$, then $P_2=\emptyset$, and $clconv(P)=P_1$. To eliminate these trivial cases, we impose the following assumption throughout the rest of the paper.

Assumption 1. $I_+^1\ne\emptyset, I_+^2\ne\emptyset$.

Similarly, if $(a_0,a)=\lambda(b_0,b)$ for some $\lambda>0$ then the two constraints are scaled version of each other. We conclude that $clconv(P)=P_1=P_2$. For this reason, we impose the following assumption.

Assumption 2. $(a_0,a)\ne\lambda(b_0,b), \forall\lambda>0$

In the following proposition, we show that $clconv(P)$ is a polyhedron and give some characteristics of its extreme points and extreme rays.

Proposition 3.1. *The closure convex hull of P is a polyhedron. Further, extreme points of $clconv(P)$ are of the form $\frac{a_0}{a_i}e_i$ for $i \in I_+^1$ and $\frac{b_0}{b_i}e_i$ for $i \in I_+^2$ Extreme rays of $clconv(P)$ are of the form $\frac{a_0}{a_i}e_i - \frac{a_0}{a_j}e_j$, for $i \in I_+^1, j \in I_-^1$, e_k for $k \in I_0^1$, and $\frac{b_0}{b_i}e_i - \frac{b_0}{b_j}e_j$, for $i \in I_+^2, j \in I_-^2$, e_k for $k \in I_0^2$.*

Proof. It is clear that $clconv(P)$ can be expressed as the projection of a polyhedron in higher dimension using disjunctive programming. Because the projection of a polyhedron is a polyhedron, it follows that $clconv(P)$ is a polyhedron. Observe further that the extreme points of P_1 are $\frac{a_0}{a_i}e_i$ for $i \in I_+^1$, and the extreme rays of P_1 are $\frac{a_0}{a_i}e_i - \frac{a_0}{a_j}e_j$, for $i \in I_+^1, j \in I_-^1$, e_k for $k \in I_0^1$. Similarly, the extreme points of P_2 are $\frac{b_0}{b_i}e_i$ for $i \in I_+^2$, and the extreme rays of P_2 are $\frac{b_0}{b_i}e_i - \frac{b_0}{b_j}e_j$, for $i \in I_+^2, j \in I_-^2$, e_k for $k \in I_0^2$. The result follows since the extreme points and the extreme rays of $clconv(P)$ are either extreme points or extreme rays of either P_1 or P_2. We emphasize that, even though Proposition 1 describes all the vectors that can possibly be extreme points and extreme rays for $clconv(P)$ some of the vectors listed might not be extreme for specific instances. Next, we argue that under the above assumptions cl $conv(P)$ is full-dimensional.

Proposition 2. $\dim(clconv(P)) = n$.

Although, to the best of our knowledge, P has not been formally studied in the past, several families of valid inequalities have been developed for its closure convex hull. We review these results next.

Proposition 3. *(Ibaraki(1973)) The C-cut*

$$\sum_{j \in I_+^1 \cup I_+^2} \max\left\{\frac{a_j}{a_0}, \frac{b_j}{b_0}\right\} x_j \geq 1 \qquad (4)$$

is valid for $clconv(P)$.

To keep the discussion self-contained, and to provide motivation for the cut-derivation strategy we propose later, we next derive (4) using classical disjunctive arguments. In particular, we use the following result.

Lemma 2.(Balas (1985)*) Let S_1 and S_2 be subsets of R_+^n. If $\sum_{j \in N} \pi_j^1 x_j \geq \pi_0^1$ is valid for S_1 and $\sum_{j \in N} \pi_j^2 x_j \geq \pi_0^2$ is valid for S_2, then $\sum_{j \in N} \max\{\pi_j^1, \pi_j^2\} x_j \geq \min\{\pi_0^1, \pi_0^2\}$ is valid for $S_1 \cup S_2$.*

We observe that, when P_1 and P_2 are polyhedra, all valid inequalities for $clconv(P)$ can be obtained using the procedure described in Lemma 2. We now give a derivation for (4) using Lemma 2. When $y = 0$, P_1 is defined by $\sum_{i \in I_+^1} a_j x_j = a_0 - \sum_{j \in N \setminus I_+^1} a_j x_j$ Therefore, the inequality $\sum_{i \in I_+^1} \frac{a_j}{a_0} x_j \geq 1$ is valid for P_1. Similarly, we can establish that $\sum_{i \in I_+^2} \frac{b_j}{b_0} x_j \geq 1$ is valid for P_2. Applying Lemma 2 we conclude that (4) is valid for $P_1 \cup P_2$. An improvement to the C-cut was proposed by various authors, as we described next.

Proposition 4. *The simple cut*

$$\sum_{j \in N} \max\left\{\frac{a_j}{a_0}, \frac{b_j}{b_0}\right\} x_j \geq 1 \qquad (5)$$

is valid for $clconv(P)$.

Judice et al (2006) refer to (5) as *disjunctive cut for basic solutions*. Again, this inequality can be derived using Lemma 2 as follows. Imposing the disjunction that $y \leq 0$ or $z \leq 0$, we conclude that $\sum_{j \in N} \frac{a_j}{a_0} \geq 1$ is valid for P_1, and $\sum_{j \in N} \frac{b_j}{b_0} \geq 1$ is valid for P_2. It follows from Lemma 2 that the disjunctive cut (5) is valid for $clconv(P)$. It is clear that the simple cut dominates the C-cut since (4) and (5) have the same positive coefficients but all negative coefficients in (5) are replaced by zeros in (4). Although, the C-cut and the simple cut are valid for $clconv(P)$, they are not sufficient to describe $clconv(P)$. In fact, they may not even be facet-defining for $clconv(P)$. We illustrate these two observations in the following example.

Example 1. *Consider the instance of P defined by*

$$y = 1 - x_1 + \frac{1}{2}x_2 + x_3$$

$$z = 1 - \frac{2}{3}x_1 + x_2 + \frac{1}{3}x_3$$

For this set, the C-cut is $x_1 \geq 1$ and the simple cut is $x_1 - \frac{1}{2}x_2 - \frac{1}{3}x_3$. We now prove that the nequality $x_1 - \frac{1}{2}x_2 - \frac{1}{2}x_3$ is an inequality which dominates both the C-cut and the simple cut, is valid for P. Observe that $x_1 - \frac{1}{2}x_2 - x_3$ is valid for P_1.

Similarly, using a scaling factor of $\frac{3}{2}$, *we can see that* $x_1 - \frac{3}{2}x_2 - \frac{1}{2}x_3 \ge \frac{3}{2}$ *is valid for* P_2. *Applying Lemma 2, we got the result.*

We next discuss sufficient conditions under which the C-cut and the simple cut define facets of $clconv(P)$. To streamline notation, we let $\alpha_i = \max\left\{\frac{a_i}{a_0}, \frac{b_i}{b_0}\right\}$ for $i \in N$. Further, we define

$T_0 = \{i \in N \mid \alpha_i = 0\}$,

$T_+ = \{i \in N \mid \alpha_i > 0\}$ and $T_- = \{i \in N \mid \alpha_i < 0\}$.

Proposition 5. *If* $|N_{++}| = 1$ *and* $\frac{a_j}{a_0} = \frac{b_j}{b_0}$ *for* $j \in N_{++}$ *then the simple cut is facet defining for* $clconv(P)$.

Proposition 6. *If* $|N_{--}| = 0$ *then the C-cut and the simple cut are identical. Further, the simple cut is the only facet-defining inequality for* $clconv(P)$ *that is of the form* $\sum_{j \in N} \gamma_j x_j \ge 1$.

We next use the disjunctive result of Lemma 2 to derive new cuts for $clconv(P)$. We refer to the resulting inequalities as *extension cuts*.

Proposition 7. *For* $p \in N_{++} \cup N_{--}$, *extension cuts*

$$\sum_{j \in N} \max\left\{\frac{b_p}{a_p}\frac{a_j}{b_0}, \frac{b_j}{b_0}\right\} x_j \ge 1, \quad \text{if } \frac{b_p}{a_p}\frac{a_0}{b_0} \ge 1 \qquad (6)$$

$$\sum_{j \in N} \max\left\{\frac{a_j}{a_0}, \frac{a_p}{b_p}\frac{b_j}{a_0}\right\} x_j \ge 1, \quad \text{if } \frac{a_p}{b_p}\frac{b_0}{a_0} \ge 1 \qquad (7)$$

are valid for $clconv(P)$.

Proof. For $p \in N_{++} \cup N_{--}$, we have that $\frac{a_p}{b_p} > 0$. Inequality $\sum_{j \in N} \frac{a_j}{a_0} x_j \ge 1$ is valid for P_1. If $\frac{b_p}{a_p}\frac{a_0}{b_0} \ge 1$, we multiply both sides of this inequality by $\frac{b_p}{a_p}\frac{a_0}{b_0}$ to obtain $\sum_{j \in N} \frac{b_p}{a_p}\frac{a_j}{b_0} x_j \ge \frac{b_p}{a_p}\frac{a_0}{b_0}$, which is valid for P_1. Since $\sum_{j \in N} \frac{b_j}{b_0} x_j \ge 1$ is valid for P_2, we conclude from Lemma 2 that (6) is valid for $clconv(P)$. Similarly, if $\frac{a_p}{b_p}\frac{b_0}{a_0} \ge 1$, then $\sum_{j \in N} \frac{a_p}{b_p}\frac{b_j}{a_0} x_j \ge \frac{a_p}{b_p}\frac{b_0}{a_0}$ is valid for P_2. Since $\sum_{j \in N} \frac{a_j}{a_0} x_j \ge 1$ is valid for P_1, we conclude from Lemma 2 that (7) is valid for $clconv(P)$. We next illustrate the use of Proposition 7 on the set described in Example 1, where it was shown that both the C-cut and the simple cut are weak.

Example 2. *Consider the set introduced in Example 1. We note that* $N_{++} = \{1\}$ *and* $N_{--} = \{2,3\}$. *Proposition 7 shows that extension cuts* $x_1 - \frac{1}{2}x_2 - \frac{1}{2}x_3$ *for* $p = 1$, $2x_1 - x_2 - \frac{1}{3}x_3 \ge 1$ *for* $p = 2$ *and* $2x_1 - \frac{1}{2}x_2 - x_3 \ge 1$ *for* $p = 3$ *are valid for* $clconv(P)$. *Inequality* $x_1 - \frac{1}{2}x_2 - \frac{1}{2}x_3$ *is facet-defining for* $clconv(P)$ *and dominates the simple cut.* If $\frac{a_p}{a_0} = \frac{b_p}{b_0}$, the extension cut reduces to the simple cut. We next discuss sufficient conditions under which the extension cut dominates the simple cut.

Proposition 8. *The extension cut dominates the simple cut if one of the following conditions is satisfied:*

(i) $I_+^2 = \{p\} \subseteq I_+^1$ *and* $\frac{b_p}{a_p}\frac{a_0}{b_0} \ge 1$ *for some* $p \in N$

(ii) $I_+^2 = \{p\} \subseteq I_+^1$ *and* $\frac{a_p}{b_p}\frac{b_0}{a_0} \ge 1$ *for some* $p \in N$

(iii) $|N_{++}| = 1, |N_{--}^0| = n - 1$.

In general, extension cuts do not need to be facet-defining for $clconv(P)$. Proposition 9 gives a sufficient condition for extension cuts to define facets of $clconv(P)$.

Proposition 9. *If* $|N_{++}| = 1$ *and* $|N_{--}^0| = n - 1$, *then the extension cut is the only facet-defining inequality for* $clconv(P)$ *of the form* $\sum_{j \in N} \gamma_j x_j \ge 1$.

In this section, we showed that existing tableaux cuts for LPCCs can be obtained using a simple disjunctive argument. We also showed that this simple disjunctive argument can be used to derive new and sometimes stronger inequalities. These cuts however are typically not sufficient to describe $clconv(P)$.

3. MULTIPLE COMPLEMENTARITY CONSTRAINTS

In the previous section, we describe how strong cutting planes can be derived from simplex tableaux of the LP relaxation of LPCCs using a single complementarity constraint between nonzero basic variables. In this section, we discuss how these results can be used to derive valid inequalities that take into account multiple complementarity constraints. The idea we pursue is analogous to that of how split cuts can be generated from the split disjunction $\sum_{i \in N} \pi_i x_i \le \pi_0$, or $\sum_{i \in N} \pi_i x_i \ge \pi_0 + 1$, where $(\pi, \pi_0) \in Z^{n+1}$ by aggregating multiple integer variables x_i into a single integer variable $z = \sum_{i \in N} \pi_i x_i$.

We first introduce the notions of split disjunctions and split closure for LPCCs with bounded variables. To motivate the idea, we first show on an example that a valid inequality can be obtained that considers multiple complementarity constraints using only cutting plane techniques involving a single complementarity constraint.

Example 3. *Consider set P^2 defined by the constraints*

$$
\begin{aligned}
y_1 &= 1 - \frac{1}{2}x_1 - \frac{1}{3}x_2 + \frac{1}{4}x_3 + \frac{1}{5}x_4 \\
y_2 &= 1 + x_1 - \frac{2}{3}x_2 - \frac{1}{3}x_3 + \frac{3}{4}x_4 \\
z_1 &= 1 + \frac{3}{4}x_1 - \frac{2}{3}x_2 - x_3 + x_4 \\
z_2 &= 1 - x_1 - x_2 + 2x_3 - \frac{1}{2}x_4
\end{aligned}
$$

where $(x; y_1, y_2, z_1, z_2) \in R_+^4 \times [0,1]^4$ are such that $y_1 z_1 = 0$ and $y_2 z_2 = 0$. Set P^2 viewed as the relaxation of an LPCC obtained by keeping four tableaux rows associated with two sets of basic complementarity variables. It is simple to verify that disjunction $(y_1 + y_2 \le 1) \vee (z_1 + z_2 \le 0)$ must be satisfied by all feasible solutions to P^2 . This disjunction can be written as

$$(-\frac{1}{2}x_1 + x_2 + \frac{1}{12}x_3 - \frac{19}{20}x_4 \ge 1) \vee (\frac{1}{8}x_1 + \frac{5}{6}x_2 - \frac{1}{2}x_3 - \frac{1}{4}x_4 \ge 1) \qquad (8)$$

Feasible solutions to this disjunction are also feasible solutions to the following one-complementarity corner relaxation

$$
\begin{aligned}
\bar{y} &= 1 + \frac{1}{2}x_1 - x_2 - \frac{1}{12}x_3 + \frac{19}{20}x_4 + x_5 \\
\bar{z} &= 1 - \frac{1}{8}x_1 - \frac{5}{6}x_2 + \frac{1}{2}x_3 + \frac{1}{4}x_4 + x_6
\end{aligned}
$$

where $(x; \bar{y}, \bar{z}) \in R_+^6 \times R^2$ are such that $\bar{y}\bar{z} = 0$. It then follows from Lemma 2 that the inequality

$$\max\{ua, rb\}^T x \ge \min\{ua_0, rb_0\} \qquad (9)$$

is valid for P^2 where $(a, a_0) = (-\frac{1}{2}, 1, \frac{1}{12}, -\frac{19}{20}, -1, 0, 1)$

and $(b, b_0) = (\frac{1}{8}, \frac{5}{6}, -\frac{1}{2}, -\frac{1}{4}, 0, -1, 1)$. *Among others, we obtain the nontrivial inequalities:*

$$
\begin{aligned}
&\frac{3}{20}x_1 + x_2 + \frac{1}{12}x_3 - \frac{3}{10}x_4 \ge 1 \\
&\frac{1}{8}x_1 + x_2 + \frac{1}{12}x_3 - \frac{1}{4}x_4 \ge 1 \\
&\frac{57}{120}x_1 + \frac{19}{6}x_2 + \frac{1}{12}x_3 - \frac{57}{60}x_4 \ge 1 \\
&-\frac{1}{8}x_1 + \frac{1}{4}x_2 + \frac{1}{2}x_3 + \frac{1}{4}x_4 \ge -1 \\
&\frac{5}{12}x_1 - \frac{5}{6}x_2 + \frac{1}{2}x_3 + \frac{19}{24}x_4 \ge -1 \\
&\frac{1}{2}x_1 + \frac{5}{36}x_2 - \frac{1}{12}x_3 + \frac{171}{180}x_4 \ge -1
\end{aligned}
$$

More generally, we observe that, for the above example, all inequalities will have nonnegative coefficients for variables x_5 and x_6 . Therefore when the convex hull description is projected onto the space of variables (x_1, x_2, x_3, x_4) only those inequalities that have zero coefficients for both x_5 and x_6 will remain and be necessary to describe the convex hull of solutions to (8). To describe the general procedure, we consider the rational polytope $P = \{(x, y, z) \in [0,1]^{n+2k} \mid Ax + By + Cz \ge d\}$ and the complementarity set

$$C_j = \{(x, y, z) \in [0,1]^{n+2k} \mid y_i z_i = 0, \forall i = 1,.., j\}$$

where $j \in K = \{1,.., k\}$. Because the inequalities $y_i + z_i \le 1$ for $i \in K$ are valid for the convex hull of $P^{\perp} = P \cap C_k$, we assume wlog that they are part of the formulation of P . We define the split disjunction associated with $(\pi, \pi_0) \in Z^k \times Z$ to be the disjunction defined by the constraints

$$\sum_{i \in K} \pi_i^+ y_i + \sum_{i \in K} (-\pi_i^-) z_i \le \pi_0 + \sum_{i \in K} (-\pi_i^-) \qquad (10)$$

$$\sum_{i \in K} (-\pi_i^-) y_i + \sum_{i \in K} \pi_i^+ z_i \le \sum_{i \in K} \pi_i^+ - \pi_0 - 1 \qquad (11)$$

where $\pi_i^+ = \max\{\pi_i, 0\}$ and $\pi_i^- = \min\{\pi_i, 0\}$ for $i \in K$.

Proposition 10. *The split disjunction (10)-(11) is satisfied by all $(x, y, z) \in C_k \cap R_+^n \times [0,1]^{2k}$.*

We next describe a procedure to develop cuts from multiple complementarity constraints of an LPCC using split disjunctions. For any $(\pi, \pi_0) \in Z^k \times Z$, we define

$$P^{(\pi, \pi_0)} = conv(P_1 \cup P_2)$$

where

$P_1 = P \cap \{(x, y, z) \in R^n \times [0,1]^{2k} \mid \pi^+ y - \pi^- z \le \pi_0 - \pi^- e\}$ and

$P_2 = P \cap \{(x, y, z) \in R^n \times [0,1]^{2k} \mid -\pi^- y + \pi^+ z \le \pi^+ e - \pi_0 - 1\}$

We say that $\alpha x + \beta y + \gamma z \ge \delta$ is a split cut associated with the disjunction (π, π_0) if it is valid for $P^{(\pi, \pi_0)}$. For instance, consider the relaxation of $P^{\perp}$ obtained from a simplex tableaux of its LP relaxation after keeping only rows containing complementarity basic variables. In particular, this relaxation is of the form

$$
\begin{aligned}
y_i &= a_{0i} - \sum_{j \in N} a_{ij} x_j \\
z_i &= b_{0i} - \sum_{j \in N} b_{ij} x_j \qquad (12) \\
y_i z_i &= 0, i \in K \\
x_i &\ge 0, i \in N
\end{aligned}
$$

For any $(\pi, \pi_0) \in Z^k \times Z$, define $I^+ = \{i \in N \mid \pi_i \ge 0\}$

and $I^- = \{i \in N \mid \pi_i < 0\}$.The first side (10) of the split disjunction can be written as

$$\sum_{j\in N}(\sum_{i\in I^+}\pi_i a_{ij} - \sum_{i\in I^-}\pi_i b_{ij})x_j \geq \sum_{i\in I^+}\pi_i a_{0i} + \sum_{i\in I^-}\pi_i(1-b_{0i}) - \pi_0$$

after substituting the equations for y_i and z_i given in (12). Similarly, the second side (11) of the split disjunction can be written as

$$\sum_{j\in N}(-\sum_{i\in I^-}\pi_j a_{ij} + \sum_{i\in I^+}\pi_i b_{ij})x_j \geq -\sum_{i\in I^-}\pi_i a_{0i} + \sum_{i\in I^+}\pi_i(b_{0i}-1) + \pi_0 + 1$$

As discussed in Example 3, this disjunction is that which naturally arises in a one-complementarity corner relaxation of the form

$$\begin{aligned} \bar{y} &= \theta_0 - \sum_{j\in N}\theta_j x_j + s_1 \\ \bar{z} &= \phi_0 - \sum_{j\in N}\phi_j x_j + s_2 \\ \bar{y}\bar{z} &= 0, x_j \geq 0, j \in N, s_1 \geq 0, s_2 \geq 0 \end{aligned} \tag{13}$$

where

$$\theta_j = \sum_{i\in I^+}\pi_i a_{ij} - \sum_{i\in I^-}\pi_i b_{ij}, \theta_0 = \sum_{i\in I^+}\pi_i a_{0i} + \sum_{i\in I^-}\pi_i(1-b_{0i}) - \pi_0$$

and

$$\phi_j = -\sum_{i\in I^-}\pi_j a_{ij} + \sum_{i\in I^+}\pi_i b_{ij}, \phi_0 = -\sum_{i\in I^-}\pi_i a_{0i} + \sum_{i\in I^+}\pi_i(b_{0i}-1) + \pi_0 + 1$$

and (π,π_0) is chosen such that $\theta_0 \neq 0$, and $\phi_0 \neq 0$. Using Lemma 2, inequality

$$\max\{u\theta, r\phi\}x + \max\{-u,0\}s_1 + \max\{0,-r\}s_2 \geq \min\{u\theta_0, r\phi_0\}$$

(14) is valid for $clconv(P^\perp)$ where $u, r \in R$. Further all valid inequalities for the closure convex hull of (13) can be obtained in this way. Note that in (14), the coefficients of s_1 and s_2 are nonnegative. It follows that the closure convex hull can be easily projected onto the space of x variables by simply deleting all inequalities that have positive coefficients for s_1 and s_2. In other words, one can find the closure convex hull of (13) by using weights u and r are chosen to be positive. In particular, the following cuts

$$\sum_{j\in N}\max\{\frac{\theta_j}{\theta_0}, \frac{\phi_j}{\phi_0}\}x_j \geq 1$$

$$\sum_{j\in N}\max\{\frac{\theta_j}{\theta_0}, \phi_j\frac{\theta_p}{\phi_p\theta_0}\}x_j \geq 1, if\ \frac{\theta_p\phi_0}{\phi_p\theta_0} \geq 1$$

$$\sum_{j\in N}\max\{\theta_j\frac{\phi_p}{\theta_p\phi_0}, \frac{\phi_j}{\phi_0}\}x_j \geq 1, if\ \frac{\phi_p\theta_0}{\theta_p\phi_0} \geq 1$$

for $p \in N_{++} \cup N_{--}$ which are simple cut and extension cuts for (13) respectively are valid for $clconv(P^\perp)$.

4. CONCLUSION

In this paper, we obtain the linear description for the convex hull of the one complementarity corner relaxation of an LPCC using a new E&R procedure that generates cuts in the space of the original variables. As a special case, we strengthen the well-known C-cut and simple cut that were introduced in the literature. We then use this result and the notion of split disjunction to derive cuts that simultaneously exploit multiple complementarity constraints for LPCCs with bounded variables.

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